SYMPLECTIC DISPLACEMENT ENERGY FOR EXACT LAGRANGIAN IMMERSIONS

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Abstract. We give an inequality of the displacement energy for exact Lagrangian immersions and the symplectic area of punctured holomorphic discs. Our approach is based on Floer homology for Lagrangian immersions [1] and Chekanov's homotopy technique of continuations [2].

1. Introduction

Let (M, ω) be a symplectic manifold, and $\iota : L \to M$ a Lagrangian immersion, i.e. $\iota : L \to M$ is an immersion which satisfies $\dim L = \dim M/2$ and $\iota^*\omega = 0$. We call a Lagrangian immersion $\iota : L \to M$ *exact* if

$$\int_{D^2} v^*\omega = 0$$

for any pair of smooth maps $v : D^2 \to M$ and $\bar{v} : \partial D^2 \to L$ such that $v|_{\partial D^2} = \iota \circ \bar{v}$, where $D^2 := \{z \in \mathbb{C} : |z| \leq 1\}$. Let $K(\iota)$ denote the set of the pairs of smooth maps $v : D^2 \to M$ and $\bar{v} : [0,1] \to L$ such that:

- $\bar{v}(0) \neq \bar{v}(1)$ and $\iota \circ \bar{v}(0) = \iota \circ \bar{v}(1)$,
- $v|_{\partial D^2} = \iota \circ \bar{v}$, where we identify $e^{2\pi i\theta} \in \partial D^2$ with $\theta \in [0,1]$.

Then we define σ by

$$\sigma := \inf \left\{ \int_{D^2} v^*\omega : (v, \bar{v}) \in K(\iota) \text{ with } \int_{D^2} v^*\omega > 0 \right\}.$$

Note that $\sigma = \infty$ if $\int_{D^2} v^*\omega = 0$ for any $(v, \bar{v}) \in K(\iota)$.

A smooth function $H : [0,1] \times M \to \mathbb{R}$ defines the time-dependent Hamiltonian vector field X_H on M by $dH = \omega(X_H, \cdot)$, and φ_t^H denotes the Hamiltonian isotopy generated by X_H, i.e. $\varphi_t^H : M \to M$ is given by

$$\frac{d\varphi_t^H}{dt} = X_H \circ \varphi_t^H \quad \text{and} \quad \varphi_0^H = \mathrm{id}.$$

We call φ_1^H the time one map generated by X_H. If M is non-compact, we assume that H is compactly supported. Then, following Hofer [8], we define a norm of H by

$$\|H\| := \int_0^1 \left(\max_{x \in M} H(t,x) - \min_{x \in M} H(t,x) \right) dt.$$

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Our main theorem is the following:

Theorem 1.1. *Let (M,ω) be a closed symplectic manifold or a non-compact symplectic manifold with convex end, and $\iota : L \to M$ an exact Lagrangian immersion from a closed manifold L. Suppose the non-injective points of $\iota : L \to M$ are transverse. Let φ_1^H be the time one map generated by X_H. If $\|H\| < \sigma$, and $\iota : L \to M$ and $\varphi_1^H \circ \iota : L \to M$ intersect transversely, then*

$$\sharp \left\{ (x,x') \in L \times L : \iota(x) = (\varphi_1^H \circ \iota)(x') \right\} \geq \sum_{k=0}^{\dim L} \dim H_k(L;\mathbb{Z}_2).$$

Since we use pseudoholomorphic curves, we put the convex end condition when M is non-compact; for example, the cotangent bundles of closed manifolds equipped with the canonical symplectic structure, the symplectic vector spaces $\mathbb{C}^n$ and so on.

Following Hofer [9], we define the *symplectic displacement energy* $e(A)$ for a subset $A \subset M$ by

$$e(A) := \inf \left\{ \|H\| : \begin{array}{l} A \cap \varphi_1^H(A) = \emptyset, \text{ where } \varphi_1^H \text{ is the time one} \\ \text{map generated by } X_H \end{array} \right\}.$$

Note that $e(A) = \infty$ if $A \cap \varphi_1^H(A) \neq \emptyset$ for any φ_1^H.

Since we may perturb H to be generic so that $\iota : L \to M$ and $\varphi_1^H \circ \iota : L \to M$ intersect transversely, we obtain the following corollary:

Corollary 1.2. *For our exact Lagrangian immersion $\iota : L \to M$,*

$$\sigma \leq e(\iota(L)).$$

The symplectic displacement energy for Lagrangian *submanifolds* was early discussed by Polterovich [13], Chekanov [2], [3] and Oh [12]. We call a Lagrangian submanifold $L \subset M$ *rational* if

$$\left\{ \int_{D^2} v^*\omega : v(D^2, \partial D^2) \to (M,L) \right\} = \Sigma\mathbb{Z}$$

for some $\Sigma > 0$. Polterovich [13] proved the following theorem; his approach is based on Gromov's theory of pseudoholomorphic curves:

Theorem 1.3. *For a rational Lagrangian submanifold $L \subset M$,*

$$\frac{1}{2}\Sigma \leq e(L).$$

Moreover, Chekanov [2] improved the Polterovich's theorem:

Theorem 1.4. *For a rational Lagrangian submanifold $L \subset M$,*

$$\Sigma \leq e(L).$$

In fact, he introduced a variant of Floer homology and obtained Theorem 1.4 as a corollary of the following theorem ([2]):

Theorem 1.5. *Let $L \subset M$ be a rational Lagrangian submanifold. If $\|H\| < \Sigma$, and L and $\varphi_1^H(L)$ intersect transversely, then*

$$\sharp \left(L \cap \varphi_1^H(L) \right) \geq \sum_{k=0}^{\dim L} H_k(L; \mathbb{Z}_2).$$

After that, Chekanov [3] introduced some homological algebra and relaxed the assumption of Theorem 1.5; and Oh [12] used Gromov–Floer theory of perturbed Cauchy–Riemann equation and simplified the proof of the inequality for the symplectic displacement energy of Lagrangian submanifolds given in [3].

We observe Corollary 1.2. Let $(M,\omega) = (\mathbb{R}^2, dx\wedge dy)$ be the 2-dimensional symplectic vector space, and $\iota : L := \{e^{i\theta} : 0 \leq \theta \leq 2\pi\} \to \mathbb{R}^2$, $e^{i\theta} \mapsto (\sin\theta\cos\theta, \sin\theta)$, an exact Lagrangian immersion "*figure 8*." Then $\sigma = \frac{2}{3}$. On the other hand, from the Hofer–Zehnder capacity [10], $e(\iota(L)) \geq \frac{4}{3}(= 2\sigma)$; and moreover, the following H attains $e(\iota(L))$ and its Hofer norm is $\frac{4}{3}$:

$$H(x,y) := \begin{cases} 0 & y \leq -1, \\ -\int_{-1}^{y} 2t\sqrt{1-t^2}dt & -1 \leq y \leq 0, \\ \frac{2}{3} + \int_0^y 2t\sqrt{1-t^2}dt & 0 \leq y \leq 1, \\ \frac{4}{3} & 1 \leq y. \end{cases}$$

(Cut H outside of a large disc to be compactly supported.) Thus $\sigma = \frac{2}{3} < e(\iota(L)) = \frac{4}{3}$. The author does not know any non-trivial example which attains the equality of Corollary 1.2.

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2. **Morse theory for Floer homology of exact Lagrangian immersions**

We prepare some notation and review Morse theory for our Floer homology of exact Lagrangian immersions.

Let (M,ω) be a symplectic manifold, and $\iota : L \to M$ an exact Lagrangian immersion; and let $H = H(t,x) : [0,1] \times M \to \mathbb{R}$ be a smooth function. Fix $s \in (0,1]$; and X_{sH} is the time-t dependent Hamiltonian vector field on M given by $d(sH) = \omega(X_{sH}, \cdot)$. Note that $X_{sH} = sX_H$. Let $\varphi_t^{sH} : M \to M$ be the Hamiltonian isotopy generated by X_{sH}, i.e. $\varphi_t^{sH} : M \to M$ is given by

$$\frac{d\varphi_t^{sH}}{dt} = X_{sH} \circ \varphi_t^{sH} \quad \text{and} \quad \varphi_0^{sH} = \mathrm{id}.$$

Fix a point $x_0 \in L$. We define Ω to be the set of the pairs of smooth maps $\gamma : [0,1] \to M$ and $\bar\gamma : \{0,1\} \to L$ such that:

- $\gamma(0) = \iota \circ \bar\gamma(0)$ and $\gamma(1) = \iota \circ \bar\gamma(1)$,

- there is a pair of smooth maps $u : [0,1] \times [0,1] \to M$ and $\bar{u} : [0,1] \times \{0,1\} \to L$ such that:
  - $u(\tau, 0) = \iota \circ \bar{u}(\tau, 0)$ and $u(\tau, 1) = \iota \circ \bar{u}(\tau, 1)$,
  - $u(0,t) = \iota(x_0)$, and $\bar{u}(0,0) = x_0$ and $\bar{u}(0,1) = x_0$,
  - $u(u,t) = \gamma(t)$, and $\bar{u}(1,0) = \bar{\gamma}(0)$ and $\bar{u}(1,1) = \bar{\gamma}(1)$.

Let $\boldsymbol{\gamma}$ denote $(\gamma, \bar{\gamma}) \in \Omega$. We may think of the tangent space $T_{\boldsymbol{\gamma}}\Omega$ as the set of the triples of a section ξ of γ^*TM, $\xi_0 \in T_{\bar{\gamma}(0)}L$ and $\xi_1 \in T_{\bar{\gamma}(1)}L$ such that $\xi(0) = \iota_*\xi_0$ and $\xi(1) = \iota_*\xi_1$; for simplicity, we omit to write the $T_{\bar{\gamma}(0)}L$ and $T_{\bar{\gamma}(1)}L$ components of $T_{\boldsymbol{\gamma}}\Omega$.

We define a functional $F_s : \Omega \to \mathbb{R}$ by

$$F_s(\boldsymbol{\gamma}) := -\int_{[0,1]\times[0,1]} u^*\omega - s\int_0^1 H(t, \gamma(t))dt.$$

Since $\iota : L \to M$ is exact Lagrangian, F_s is independent of the choice of $(u, \bar{u})$, i.e. F_s depends only on $\boldsymbol{\gamma}$. The differential dF_s is given by

$$(dF_s)_{\boldsymbol{\gamma}}(\xi) = \int_0^1 \omega\left(\xi(t), -\frac{d\gamma(t)}{dt} + sX_H(\gamma(t))\right)dt$$

for $\xi \in T_{\boldsymbol{\gamma}}\Omega$. Hence $\boldsymbol{\gamma}$ is a critical point of F_s if and only if γ is a time-1 trajectory of X_{sH} which starts and ends on $\iota(L)$. We define c_s to be the set of the critical points of F_s, i.e.

$$c_s := \left\{\boldsymbol{\gamma} := (\gamma, \bar{\gamma}) \in \Omega : \frac{d\gamma(t)}{dt} = X_{sH}(\gamma(t))\right\}.$$

Note that, let $\gamma(t) := \varphi_t^{sH}(\delta(t))$, then $(\gamma, \bar{\gamma}) \in c_s$ if and only if

$$\delta(t) \equiv p \in \iota(L) \cap (\varphi_1^{sH})^{-1}(\iota(L)).$$

Thus $\boldsymbol{\gamma} \in c_s$ gives an intersection point of $\iota(L)$ and $(\varphi_1^{sH})^{-1}(\iota(L))$.

Let $\{J_t\}_{t\in[0,1]}$ be a time-dependent tame almost complex structure on M. We define a Riemannian metric G on Ω by

$$G(\xi_1, \xi_2) := \int_0^1 \omega(\xi_1(t), J_t\xi_2(t))dt$$

for $\xi_1, \xi_2 \in T_{\boldsymbol{\gamma}}\Omega$. Then the gradient vector of F_s with respect to G is given by

$$(\nabla F_s)_{\boldsymbol{\gamma}} = J_t(\gamma(t))\left(\frac{d\gamma(t)}{dt} - sX_H(\gamma(t))\right).$$

Note that Ω and F_s are essentially the same as used in Chekanov [3] and Oh [12] but we modify them for exact Lagrangian immersions.

3. Floer homology for exact Lagrangian immersions

We introduce a variant of Floer homology, inspired by Chekanov [2] and [3], for exact Lagrangian immersions. In this section we do *not* use $\|H\|$.

Let (M, ω) be a closed symplectic manifold or a non-compact symplectic manifold with convex end, and $\iota : L \to M$ an exact Lagrangian immersion from a closed manifold L. For generic H, there exists an open dense subset

$T \subset [0,1]$ such that, for $s \in T$, $(\varphi_1^{sH} \circ \iota)_* T_{\bar{\gamma}(0)} L$ and $\iota_* T_{\bar{\gamma}(1)} L$ intersect transversely in $T_{\gamma(1)} M$ for $(\gamma, \bar{\gamma}) \in c_s$; we always assume that H is generic and take such $T \subset [0,1]$. Note that c_s is a finite set for $s \in T$.

Let $s \in T$. We define $\mathcal{M}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\gamma} := (\gamma, \bar{\gamma}), \boldsymbol{\delta} := (\delta, \bar{\delta}) \in c_s$ to be the set of the *(descending) gradient trajectories* $(u, \bar{u})$ of F_s from $\boldsymbol{\gamma}$ to $\boldsymbol{\delta}$, i.e. the pairs of smooth maps $u : \mathbb{R} \times [0,1] \to M$ and $\bar{u} : \mathbb{R} \times \{0,1\} \to L$ such that:

- $u(\tau, 0) = \iota \circ \bar{u}(\tau, 0)$ and $u(\tau, 1) = \iota \circ \bar{u}(\tau, 1)$,
- $\lim_{\tau \to -\infty} u(\tau, t) = \gamma(t)$ and $\lim_{\tau \to -\infty} \bar{u}(\tau, i) = \bar{\gamma}(i)$ for $i = 0, 1$,
- $\lim_{\tau \to \infty} u(\tau, t) = \delta(t)$, and $\lim_{\tau \to \infty} \bar{u}(\tau, i) = \bar{\delta}(i)$ for $i = 0, 1$,
- u is a solution of the perturbed Cauchy–Riemann equation:

$$\frac{\partial u(\tau, t)}{\partial \tau} + J_t(u(\tau, t)) \left(\frac{\partial u(\tau, t)}{\partial t} - s X_H(u(\tau, t)) \right) = 0.$$

Note that $\mathbb{R}$ acts on $\mathcal{M}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$ by the translations of τ, and let $\hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$ denote the quotient. Since the boundary value $\iota \circ \bar{u}$ does not switch sheets at non-injective points of the immersion, we can use the usual local theory of the perturbed Cauchy–Riemann equation. Hence we have the following theorem ([1], [4], [5] and [12]):

Theorem 3.1. *For generic* $\{J_t\}_{t \in [0,1]}$, $\hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$ *is a finite dimensional smooth manifold.*

Let $\hat{\mathcal{M}}_s^k(\boldsymbol{\gamma}, \boldsymbol{\delta})$ denote the k-dimensional component of $\hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

Following [12] and [14], we define the energy $E(u)$ of $(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$ by

$$\begin{aligned} E(u) &:= \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial u(\tau, t)}{\partial \tau} \right|_{J_t}^2 dt d\tau \\ &= \int_{-\infty}^{\infty} \int_0^1 \omega \left(\frac{\partial u(\tau, t)}{\partial \tau}, J_t(u(\tau, t)) \frac{\partial u(\tau, t)}{\partial \tau} \right) dt d\tau. \end{aligned}$$

Lemma 3.2. *For* $(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$,

$$E(u) = F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}).$$

Proof. Since u satisfies the perturbed Cauchy–Riemann equation,

$$\begin{aligned} E(u) &= \int_{-\infty}^{\infty} \int_0^1 \omega \left(\frac{\partial u(\tau, t)}{\partial \tau}, \frac{\partial u(\tau, t)}{\partial t} - s X_H(u(\tau, t)) \right) \\ &= \int_{\mathbb{R} \times [0,1]} u^* \omega + s \int_{-\infty}^{\infty} \int_0^1 \frac{\partial H(t, u(\tau, t))}{\partial \tau} dt d\tau \\ &= \int_{\mathbb{R} \times [0,1]} u^* \omega + s \int_0^1 H(t, \delta(t)) dt - s \int_0^1 H(t, \gamma(t)) dt \\ &= F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}). \end{aligned}$$

∎

Suppose $\sigma > 0$. We take $0 < \kappa < \sigma$, and choose an interval $[b_-, b_+) \subset \mathbb{R}$ such that

$$b_- < 0 < b_+ \quad \text{and} \quad b_+ - b_- = \kappa.$$

Then we define a function $f_s : c_s \to [b_-, b_+)$ by

$$f_s(\boldsymbol{\gamma}) \equiv F_s(\boldsymbol{\gamma}) \mod \kappa$$

for $\boldsymbol{\gamma} \in c_s$. Following Chekanov [2], we call $(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$ a *distinguished* gradient trajectory if

$$F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta}).$$

We define $\hat{\mathcal{M}}_s^d(\boldsymbol{\gamma}, \boldsymbol{\delta})$ to be the set of the distinguished gradient trajectories in $\hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$; in fact, since $\iota : L \to M$ is exact Lagrangian,

$$\hat{\mathcal{M}}_s^d(\boldsymbol{\gamma}, \boldsymbol{\delta}) = \begin{cases} \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta}) & \text{if } F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta}), \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\hat{\mathcal{M}}_s^{d,k}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ denote the k-dimensional component of $\hat{\mathcal{M}}_s^d(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

From Lemma 3.2, we have the following lemma:

Lemma 3.3. *For $(u, \bar{u}) \in \hat{\mathcal{M}}_s^d(\boldsymbol{\gamma}, \boldsymbol{\delta})$,*

$$E(u) < \kappa.$$

Assume that the non-injective points of $\iota : L \to M$ are transverse.

Proposition 3.4. *Let $s_i \to s_\infty \in \{0\} \cup T$, and $(u_i, \bar{u}_i) \in \hat{\mathcal{M}}_{s_i}(\boldsymbol{\gamma}_i, \boldsymbol{\delta}_i)$ be a sequence of gradient trajectories with $E(u_i) < \kappa$. Then $\{(u_i, \bar{u}_i)\}$ has a subsequence which converges to a broken gradient trajectory without bubble tree in the sense of Floer–Gromov convergence.*

Proof. Since L is compact, taking a subsequence if necessary, $\boldsymbol{\gamma}_i$ and $\boldsymbol{\delta}_i$ converge to $\boldsymbol{\gamma} \in c_{s_\infty}$ and $\boldsymbol{\delta} \in c_{s_\infty}$, respectively. Then, by Lemma 3.2, $E(u_i)$ is uniformly bounded, and the Floer–Gromov compactness theorem ([4], [5] and [14]) implies that $\{(u_i, \bar{u}_i)\}$ has a subsequence which converges to a broken gradient trajectory $((v_1, \bar{v}_1), \ldots, (v_N, \bar{v}_N)) \in \hat{\mathcal{M}}_{s_\infty}(\boldsymbol{\gamma}, \boldsymbol{\theta}_1) \times \cdots \times \hat{\mathcal{M}}_{s_\infty}(\boldsymbol{\theta}_{N-1}, \boldsymbol{\delta})$ with bubble trees; the *tail* components of the bubble trees are

(i) a pseudoholomorphic sphere $v : S^2 \to M$,

(ii) a pseudoholomorphic disc $v : D^2 \to M$ with $\bar{v} : \partial D^2 \to L$ such that $v|_{\partial D^2} = \iota \circ \bar{v}$,

(iii) a pseudoholomorphic disc $v : D^2 \to M$ of $(v, \bar{v}) \in K(\iota)$.

But, since our Lagrangian immersion is exact, the bubbles of (i) and (ii) can not occur. Moreover, since the symplectic area of the bubble trees is less than or equal to $\limsup E(u_i) \leq \kappa \ (< \sigma)$ ([14]), the bubbles of (iii) can not occur. Hence there is no bubble tree, and the subsequence converges to the broken gradient trajectory. □

To define our Floer homology, we use the following compactness theorems:

Theorem 3.5. *$\hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ is compact.*

Proof. Suppose on the contrary that $\hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\delta})$ is not compact. Then there exists a sequence $\{(u_i,\bar{u}_i)\} \subset \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\delta})$ such that any subsequence does not converge in $\hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\delta})$. On the other hand, from Lemma 3.3 and Proposition 3.4, taking a subsequence if necessary, $\{(u_i,\bar{u}_i)\}$ converges to a broken gradient trajectory $((v_1,\bar{v}_1),\dots,(v_N,\bar{v}_N))$ without bubble tree. In this case, N turns out to be 1 and $(v_1,\bar{v}_1) \in \hat{\mathcal{M}}_s^0(\boldsymbol{\gamma},\boldsymbol{\delta})$ for generic $\{J_t\}_{t\in[0,1]}$ because of the virtual dimension counting. Since the subsequence preserves the condition $F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta})$, the limit $(v_1,\bar{v}_1)$ is in $\hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\delta})$, which contradicts that any subsequence does not converge in $\hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\delta})$. Thus $\hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\delta})$ is compact. $\square$

Theorem 3.6. *$\hat{\mathcal{M}}_s^{d,1}(\boldsymbol{\gamma},\boldsymbol{\delta})$ has a suitable compactification whose boundary is given by*

$$\bigcup_{\boldsymbol{\theta}\in c_s} \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\theta},\boldsymbol{\delta}).$$

Proof. The proof is based on the standard gluing-compactness argument in [4], [5] and [14]. For a pair $((u_1,\bar{u}_1),(u_2,\bar{u}_2)) \in \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\theta},\boldsymbol{\delta})$, the gluing procedure gives a unique connected component of $\hat{\mathcal{M}}_s^1(\boldsymbol{\gamma},\boldsymbol{\delta})$, say $\mathcal{M}$, such that the pair $((u_1,\bar{u}_1),(u_2,\bar{u}_2))$ is a compactifying point of $\mathcal{M}$. Since

$$\begin{aligned} F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) &= F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta}) + F_s(\boldsymbol{\theta}) - F_s(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\theta}) + f_s(\boldsymbol{\theta}) - f_s(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta}), \end{aligned}$$

$\mathcal{M}$ is contained in $\hat{\mathcal{M}}_s^{d,1}(\boldsymbol{\gamma},\boldsymbol{\delta})$.

On the other hand, from Lemma 3.3 and Proposition 3.4, let $\{(u_i,\bar{u}_i)\} \subset \hat{\mathcal{M}}_s^{d,1}(\boldsymbol{\gamma},\boldsymbol{\delta})$ be a sequence which converges to a broken gradient trajectory $((v_1,\bar{v}_1),\dots,(v_N,\bar{v}_N))$ without bubble tree. In this case, N turns out to be 2 and $((v_1,\bar{v}_1),(v_2,\bar{v}_2)) \in \hat{\mathcal{M}}_s^0(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \hat{\mathcal{M}}_s^0(\boldsymbol{\theta},\boldsymbol{\delta})$ for generic $\{J_t\}_{t\in[0,1]}$ because of the virtual dimension counting. Since $F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta})$, there exists $m \in \mathbb{Z}$ such that

$$F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\theta}) + m\kappa, \tag{1}$$

$$F_s(\boldsymbol{\theta}) - F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\theta}) - f_s(\boldsymbol{\delta}) - m\kappa. \tag{2}$$

From (1), since $0 < E(u_1) = F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta})$ and $f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\theta}) < \kappa$, we obtain $0 \le m$; and from (2), since $0 < E(u_2) = F_s(\boldsymbol{\theta}) - F_s(\boldsymbol{\delta})$ and $f_s(\boldsymbol{\theta}) - f_s(\boldsymbol{\delta}) < \kappa$, we obtain $m \le 0$. Thus $m = 0$, which implies $((v_1,\bar{v}_1),(v_2,\bar{v}_2)) \in \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\theta},\boldsymbol{\delta})$. We obtain the compactification of $\hat{\mathcal{M}}_s^{d,1}(\boldsymbol{\gamma},\boldsymbol{\delta})$. $\square$

Analogous to the Floer homology of Lagrangian submanifolds ([4]), Theorem 3.4 allows us to define our Floer complex. Let C_s be the free $\mathbb{Z}_2$-module

$$C_s := \bigoplus_{\boldsymbol{\gamma}\in c_s} \mathbb{Z}_2\boldsymbol{\gamma}$$

and we define a linear map $\partial_s : C_s \to C_s$ by

$$\partial_s \boldsymbol{\gamma} := \sum_{\boldsymbol{\delta} \in c_s} \sharp \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma}, \boldsymbol{\delta}) \boldsymbol{\delta}$$

for $\boldsymbol{\gamma} \in c_s$. Then Theorem 3.6 implies the following theorem ([4]):

Theorem 3.7. $\partial_s \circ \partial_s = 0$.

Let $H(C_s, \partial_s)$ denote the homology of (C_s, ∂_s), which is our *distinguished* Floer homology for exact Lagrangian immersions.

Next, we prepare some notation. Let 0_L be the zero section of T^*L, and we fix a diffeomorphism $i_L : L \to 0_L$. Take a small tubular neighborhood U of 0_L in T^*L and an immersion $\pi : U \to M$ such that:

- $\pi \circ i_L = \iota$,
- $\pi^*\omega$ equals the canonical symplectic form on T^*L.

We define a smooth function $sH \circ \pi : [0,1] \times U \to \mathbb{R}$ by $(sH \circ \pi)(t, x) := sH(t, \pi(x))$. Take s to be small so that for any $(\gamma, \bar{\gamma}) \in c_s$ the image of γ is contained in $\pi(U)$. Then we divide c_s into the following two sets a_s and b_s:

$$a_s := \left\{ (\gamma, \bar{\gamma}) \in c_s : \begin{array}{l} \text{there exists } \alpha : [0,1] \to U \text{ such that } \alpha(0), \alpha(1) \in 0_L \\ \text{and } \gamma = \pi \circ \alpha, \text{ and } \alpha(i) = i_L(\bar{\gamma}(i)) \text{ for } i = 0, 1 \end{array} \right\},$$

and $b_s := \{(\gamma, \bar{\gamma}) \in c_s : (\gamma, \bar{\gamma}) \notin a_s\}$. Note that, for small s, $\bar{\gamma}(1)$ and $\bar{\gamma}(0)$ are very close in L for $(\gamma, \bar{\gamma}) \in a_s$; on the other hand, for $(\gamma, \bar{\gamma}) \in b_s$, there exists $(y, y') \in L \times L$ such that $y \neq y'$, $\iota(y) = \iota(y')$ and $(\bar{\gamma}(0), \bar{\gamma}(1))$ is very close to (y, y') in $L \times L$.

Theorem 3.8. *There exists s_0 such that, for $s < s_0$, $\{(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta}) : E(u) < \kappa\} = \emptyset$ for any $\boldsymbol{\gamma} \in a_s$ and $\boldsymbol{\delta} \in b_s$.*

Proof. Suppose on the contrary that there is no such s_0. Then there exists a sequence $s_i \to 0$ such that $\{(u, \bar{u}) \in \hat{\mathcal{M}}_{s_i}(\boldsymbol{\gamma}_i, \boldsymbol{\delta}_i) : E(u) < \kappa\} \neq \emptyset$ for some $\boldsymbol{\gamma}_i := (\gamma_i, \bar{\gamma}_i) \in a_{s_i}$ and $\boldsymbol{\delta}_i := (\delta_i, \bar{\delta}_i) \in b_{s_i}$; moreover, taking a subsequence if necessary, there exist $(x, x), (y, y') \in L \times L$ such that:

- $(\bar{\gamma}_i(0), \bar{\gamma}_i(1)) \to (x, x)$ and $\gamma_i(t) \to \iota(x)$,
- $y \neq y'$ and $\iota(y) = \iota(y')$, and $(\bar{\delta}_i(0), \bar{\delta}_i(1)) \to (y, y')$ and $\delta_i(t) \to \iota(y) = \iota(y')$.

Let $(u_i, \bar{u}_i) \in \hat{\mathcal{M}}_{s_i}(\boldsymbol{\gamma}_i, \boldsymbol{\delta}_i)$ with $E(u_i) < \kappa$. From Proposition 3.4, taking a subsequence if necessary, $(u_i, \bar{u}_i)$ converges to a broken *pseudoholomorphic strip* $((v_1, \bar{v}_1), \ldots, (v_N, \bar{v}_N))$ without bubble tree. Since (x, x) is an injective point and (y, y') is a non-injective point, at least one of the broken components $(v_i, \bar{v}_i)$ is an element of $K(\iota)$. But, since the symplectic area of the broken pseudoholomorphic strip is less than or equal to $\limsup E(u_i) \leq \kappa$ $(< \sigma)$, there is no such v_i, which contradicts to the existence of such $s_i \to 0$. Thus there exists s_0 such that, for $s < s_0$, $\{(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta}) : E(u) < \kappa\} = \emptyset$ for any $\boldsymbol{\gamma} \in a_s$ and $\boldsymbol{\delta} \in b_s$. □

From Lemma 3.3, we obtain the following corollary:

Corollary 3.9. *There exists* s_0 *such that, for* $s < s_0$, $\hat{\mathcal{M}}^d_s(\boldsymbol{\gamma}, \boldsymbol{\delta}) = \emptyset$ *for any* $\boldsymbol{\gamma} \in a_s$ *and* $\boldsymbol{\delta} \in b_s$.

Similarly, we can prove the following theorem and corollary:

Theorem 3.10. *There exists* s_0 *such that, for* $s < s_0$, $\{(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta}) : E(u) < \kappa\} = \emptyset$ *for any* $\boldsymbol{\gamma} \in b_s$ *and* $\boldsymbol{\delta} \in a_s$.

Corollary 3.11. *There exists* s_0 *such that, for* $s < s_0$, $\hat{\mathcal{M}}^d_s(\boldsymbol{\gamma}, \boldsymbol{\delta}) = \emptyset$ *for any* $\boldsymbol{\gamma} \in b_s$ *and* $\boldsymbol{\delta} \in a_s$.

Now we define the free $\mathbb{Z}_2$-modules

$$A_s := \bigoplus_{\boldsymbol{\gamma} \in a_s} \mathbb{Z}_2 \boldsymbol{\gamma} \quad \text{and} \quad B_s := \bigoplus_{\boldsymbol{\gamma} \in b_s} \mathbb{Z}_2 \boldsymbol{\gamma}.$$

Note that $C_s = A_s \oplus B_s$. From Corollary 3.9 and 3.11, the boundary operator ∂_s has no cross term when $s < s_0$, and we obtain the following corollary:

Corollary 3.12. *There exists* s_0 *such that, for* $s < s_0$, $\partial_s A_s \subset A_s$ *and* $\partial_s B_s \subset B_s$.

For $s < s_0$, let $H(A_s, \partial_s)$ and $H(B_s, \partial_s)$ denote the homologies of (A_s, ∂_s) and (B_s, ∂_s), respectively.

Since L is compact, there exists a function $\varepsilon : [0,1] \to [0, \infty)$ such that $\lim_{s \to 0} \varepsilon(s) = 0$ and $|F_s(\boldsymbol{\gamma})| < \varepsilon(s)$ for $\boldsymbol{\gamma} \in a_s$. In particular, $F_s(\boldsymbol{\gamma}) = f_s(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma} \in a_s$ when $\varepsilon(s) \leq \min\{b_+, -b_-\}$. We often use this $\varepsilon : [0,1] \to [0, \infty)$.

Proposition 3.13. *There exists* s_0 *such that for* $s < s_0$ *the image of* u *is contained in* $\pi(U)$ *for any* $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ *and* $(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

Proof. First, there exist s_0 and C such that, for $s < s_0$, $\sup_{(\tau,t)} |du(\tau, t)|_{J_t} < C$ for any $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ and $(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$. (Suppose on the contrary that there is no such s_0 nor C. Then there exist sequences $s_i \to 0$, $(u_i, \bar{u}_i) \in \hat{\mathcal{M}}_{s_i}(\boldsymbol{\gamma}_i, \boldsymbol{\delta}_i)$ for some $\boldsymbol{\gamma}_i, \boldsymbol{\delta}_i \in a_{s_i}$, and (τ_i, t_i) such that $|du_i(\tau_i, t_i)| \to \infty$. Then, take a subsequence if necessary, there appear *non-trivial* bubble trees by the rescaling argument [11]. On the other hand, $E(u_i) = F_{s_i}(\boldsymbol{\gamma}_i) - F_{s_i}(\boldsymbol{\delta}_i) < 2\varepsilon(s_i) \to 0$, which contradicts that the symplectic area of the non-trivial bubble trees is greater than 0.) Let $D(z_0; r) := \{z \in \mathbb{C} : |z - z_0| < r\} \subset \mathbb{R} \times (0,1)$. Following the mean value inequality of [11], there exists $\hbar$ such that, if u is a solution of the perturbed Cauchy–Riemann equation and $\int_{D(z_0;r)} |du|^2_{J_t} < \hbar$, then $|du(z_0)|_{J_t} \leq \frac{8}{\pi r^2} \int_{D(z_0;r)} |du|^2_{J_t}$. Take s_1 such that $2\varepsilon(s) < \hbar$ for $s < s_1$. Since $E(u) < 2\varepsilon(s)$, we obtain

$$|du(z_0)|^2_{J_t} \leq \frac{8}{\pi \varepsilon(s)^{1/2}} \int_{D(z_0;\varepsilon(s)^{1/4})} |du|^2_{J_t} \leq \frac{16}{\pi} \varepsilon(s)^{1/2}$$

for any $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ and $(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Thus for $s < \min\{s_0, s_1\}$

$$\begin{aligned}\int_0^{t_0} \left|\frac{\partial u(\tau,t)}{\partial t}\right|_{J_t} dt &\leq \int_{[0,\varepsilon(s)^{1/4}]\cup[1-\varepsilon(s)^{1/4},1]} \left|\frac{\partial u(\tau,t)}{\partial t}\right|_{J_t} dt \\ &\quad + \int_{[\varepsilon(s)^{1/4},1-\varepsilon(s)^{1/4}]} \left|\frac{\partial u(\tau,t)}{\partial t}\right|_{J_t} dt \\ &\leq 2C\varepsilon(s)^{1/4} + \frac{4}{\sqrt{\pi}}(1-2\varepsilon(s)^{1/4})\varepsilon(s)^{1/4} \\ &\to 0 \text{ (as } s \to 0)\end{aligned}$$

for any $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ and $(u, \bar{u}) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$, which implies that the image of u is contained in $\pi(U)$ when s is small. □

Finally, we have the following theorem:

Theorem 3.14. *There exists s_0 such that for $s < s_0$*

$$H(A_s, \partial_s) \cong \bigoplus_{k=0}^{\dim L} H_k(L; \mathbb{Z}_2).$$

Proof. By Proposition 3.13, there exists s_0 such that for $s < s_0$ the image of u is contained in $\pi(U)$ for any $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ and $(u, \bar{u}) \in \hat{\mathcal{M}}^d_s(\boldsymbol{\gamma}, \boldsymbol{\delta})$. In this case, (A_s, ∂_s) agrees with the usual Floer complex generated by the time-1 trajectories of $X_{sH\circ\pi}$ which start and end on 0_L in T^*L. Thus $H(A_s, \partial_s)$ is isomorphic to $\bigoplus_{k=0}^{\dim L} H_k(L; \mathbb{Z}_2)$ ([1] and [6]). □

4. **Continuations**

Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth function such that for some $R > 0$ $\rho(\tau) = s_-$ when $\tau < -R$ and $\rho(\tau) = s_+$ when $\tau > R$. We call such ρ a *continuation function.* In particular, for $0 < s \leq S \leq 1$, let ρ_+ be a *non-decreasing* continuation function such that $\rho_+(\tau) = s$ when $\tau < -R$ and $\rho_+(\tau) = S$ when $\tau > R$, and ρ_- a *non-increasing* continuation function such that $\rho_-(\tau) = S$ when $\tau < -R$ and $\rho_-(\tau) = s$ when $\tau > R$.

Let $s_-, s_+ \in T$. We define $\mathcal{M}_\rho(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\gamma} = (\gamma, \bar{\gamma}) \in c_{s_-}$ and $\boldsymbol{\delta} = (\delta, \bar{\delta}) \in c_{s_+}$ to be the set of the *continuation trajectories* $(u, \bar{u})$ from $\boldsymbol{\gamma}$ to $\boldsymbol{\delta}$, i.e. the pairs of smooth maps $u : \mathbb{R} \times [0, 1] \to M$ and $\bar{u} : \mathbb{R} \times \{0, 1\} \to L$ such that:

- $u(\tau, 0) = \iota \circ \bar{u}(\tau, 0)$ and $u(\tau, 1) = \iota \circ \bar{u}(\tau, 1)$,
- $\lim_{\tau\to-\infty} u(\tau, t) = \gamma(t)$ and $\lim_{\tau\to-\infty} \bar{u}(\tau, i) = \bar{\gamma}(i)$ for $i = 0, 1$,
- $\lim_{\tau\to\infty} u(\tau, t) = \delta(t)$ and $\lim_{\tau\to\infty} \bar{u}(\tau, i) = \bar{\delta}(i)$ for $i = 0, 1$,
- u is a solution of the perturbed Cauchy–Riemann equation:
$$\frac{\partial u(\tau,t)}{\partial \tau} + J_t(u(\tau,t))\left(\frac{\partial u(\tau,t)}{\partial t} - \rho(\tau)X_H(u(\tau,t))\right) = 0.$$

In this case, $\mathbb{R}$ does *not* act on $\mathcal{M}_\rho(\boldsymbol{\gamma}, \boldsymbol{\delta})$ when $\rho(\tau)$ is not constant. Since the boundary value $\iota \circ \bar{u}$ does not switch sheets at non-injective points of the immersion, we can use the usual local theory of the perturbed Cauchy–Riemann equation. Hence we have the following theorem:

Theorem 4.1. *For generic* $\{J_t\}_{t\in[0,1]}$, $\mathcal{M}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$ *is a finite dimensional smooth manifold.*

Let $\mathcal{M}^k_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$ denote the k-dimensional component of $\mathcal{M}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$.

We define the energy $E(u)$ of $(u,\bar{u})\in\mathcal{M}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$ by

$$\begin{aligned} E(u) &:= \int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u(\tau,t)}{\partial\tau}\right|^2_{J_t} dtd\tau \\ &= \int_{-\infty}^{\infty}\int_0^1 \omega\left(\frac{\partial u(\tau,t)}{\partial\tau}, J_t(u(\tau,t))\frac{\partial u(\tau,t)}{\partial\tau}\right) dtd\tau. \end{aligned}$$

Lemma 4.2. *For* $(u,\bar{u})\in\mathcal{M}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$,

$$E(u) = F_{s_-}(\boldsymbol{\gamma}) - F_{s_+}(\boldsymbol{\delta}) - \int_{-\infty}^{\infty}\frac{d\rho(\tau)}{d\tau}\int_0^1 H(t,u(\tau,t))dtd\tau.$$

Proof. Since u satisfies the perturbed Cauchy–Riemann equation,

$$\begin{aligned} E(u) &= \int_{-\infty}^{\infty}\int_0^1 \omega\left(\frac{\partial u(\tau,t)}{\partial\tau}, \frac{\partial u(\tau,t)}{\partial t} - \rho(\tau)X_H(u(\tau,t))\right) \\ &= \int_{\mathbb{R}\times[0,1]} u^*\omega + \int_{-\infty}^{\infty}\rho(\tau)\int_0^1 \frac{\partial H(t,u(\tau,t))}{\partial\tau}dtd\tau \\ &= \int_{\mathbb{R}\times[0,1]} u^*\omega + s_+\int_0^1 H(t,\delta(t))dt - s_-\int_0^1 H(t,\gamma(t))dt \\ &\qquad - \int_{-\infty}^{\infty}\frac{d\rho(\tau)}{d\tau}\int_0^1 H(t,u(\tau,t))dtd\tau \\ &= F_{s_-}(\boldsymbol{\gamma}) - F_{s_+}(\boldsymbol{\delta}) - \int_{-\infty}^{\infty}\frac{d\rho(\tau)}{d\tau}\int_0^1 H(t,u(\tau,t))dtd\tau. \end{aligned}$$

□

We call $(u,\bar{u})\in\mathcal{M}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$ a *distinguished* continuation trajectory if

$$F_{s_-}(\boldsymbol{\gamma}) - F_{s_+}(\boldsymbol{\delta}) = f_{s_-}(\boldsymbol{\gamma}) - f_{s_+}(\boldsymbol{\delta}),$$

and define $\mathcal{M}^d_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$ to be the set of the distinguished continuation trajectories in $\mathcal{M}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$; in fact, since $\iota: L\to M$ is exact Lagrangian,

$$\mathcal{M}^d_\rho(\boldsymbol{\gamma},\boldsymbol{\delta}) = \begin{cases} \mathcal{M}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta}) & \text{if } F_{s_-}(\boldsymbol{\gamma}) - F_{s_+}(\boldsymbol{\delta}) = f_{s_-}(\boldsymbol{\gamma}) - f_{s_+}(\boldsymbol{\delta}), \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}^{d,k}_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$ denote the k-dimensional component of $\mathcal{M}^d_\rho(\boldsymbol{\gamma},\boldsymbol{\delta})$.

If M is non-compact, we assume that $H:[0,1]\times M\to\mathbb{R}$ is compactly supported. Then

$$\int_0^1 \min_{x\in M} H(t,x)dt \le 0 \le \int_0^1 \max_{x\in M} H(t,x)dt.$$

On the other hand, if M is closed, since $H(t,x)$ and $H(t,x)+c$ for $c \in \mathbb{R}$ give the same Hamiltonian vector field, we may assume

$$\int_0^1 \min_{x\in M} H(t,x)dt \le 0 \le \int_0^1 \max_{x\in M} H(t,x)dt.$$

Define a_- and a_+ by

$$a_- := -\int_0^1 \max_{x\in M} H(t,x)dt \quad \text{and} \quad a_+ := -\int_0^1 \min_{x\in M} H(t,x)dt.$$

Note that $\|H\| = a_+ - a_-$. Suppose $\|H\| < \sigma$, and we choose $\|H\| < \kappa < \sigma$ and the interval $[b_-, b_+) \subset \mathbb{R}$ such that

$$b_- < a_- \le 0 \le a_+ < b_+ \quad \text{and} \quad b_+ - b_- = \kappa.$$

Note that we use $\|H\| < \sigma$ here.

Lemma 4.3. *There exists s_0 such that for $s < s_0$*

$$E(u) < \kappa$$

for $\boldsymbol{\gamma} \in a_s$ and $(u, \bar{u}) \in \mathcal{M}^d_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

Proof. By Lemma 4.2, since ρ_+ is non-decreasing, we obtain

$$E(u) \le F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\delta}) + (S-s)a_+.$$

Note that there exists s_0 such that, for $s < s_0$, $\varepsilon(s) < b_+ - a_+$ and $|f_s(\boldsymbol{\gamma})| < \varepsilon(s)$ for $\boldsymbol{\gamma} \in a_s$. Since $(u, \bar{u})$ is distinguished and $b_- \le f_S(\boldsymbol{\delta})$, for $s < s_0$,

$$\begin{aligned} E(u) &\le F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\delta}) + (S-s)a_+ \\ &= f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\delta}) + (S-s)a_+ \\ &< \varepsilon(s) - b_- + (S-s)a_+ \\ &< b_+ - b_- = \kappa. \end{aligned}$$

□

Proposition 4.4. *There exists s_0 such that, for $s < s_0$, $\{(u_i, \bar{u}_i)\} \subset \mathcal{M}^d_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\gamma} \in a_s$ has a subsequence which converges to a broken gradient/continuation trajectory without bubble tree in the sense of Floer–Gromov convergence.*

Proof. By Lemma 4.2, $E(u_i)$ is uniformly bounded, and the Floer–Gromov compactness theorem implies that $\{(u_i, \bar{u}_i)\}$ has a subsequence which converges to a broken gradient/continuation trajectory $((v_1, \bar{v}_1), \ldots, (v_N, \bar{v}_N)) \in \hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\theta}_1) \times \cdots \times \hat{\mathcal{M}}_s(\boldsymbol{\theta}_{i-1}, \boldsymbol{\theta}_i) \times \mathcal{M}_{\rho_+}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{i+1}) \times \hat{\mathcal{M}}_S(\boldsymbol{\theta}_{i+1}, \boldsymbol{\theta}_{i+2}) \times \cdots \times \hat{\mathcal{M}}_S(\boldsymbol{\theta}_{N-1}, \boldsymbol{\delta})$ with bubble trees; the tail components of the bubble trees are

(i) a pseudoholomorphic sphere $v : S^2 \to M$,
(ii) a pseudoholomorphic disc $v : D^2 \to M$ with $\bar{v} : \partial D^2 \to L$ such that $v|_{\partial D^2} = \iota \circ \bar{v}$,
(iii) a pseudoholomorphic disc $v : D^2 \to M$ of $(v, \bar{v}) \in K(\iota)$.

But, since our Lagrangian immersion is exact, the bubbles of (i) and (ii) can not occur. Moreover, by Lemma 4.3, there exists s_0 such that for $s < s_0$ the symplectic area of the bubble trees is less than or equal to κ ($< \sigma$), and the bubbles of (iii) can not occur. Hence there is no bubble tree and the subsequence converges to the broken gradient/continuation trajectory. □

To define our continuations, we use the following compactness theorems:

Theorem 4.5. *There exists s_0 such that, for $s < s_0$, $\mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\gamma} \in a_s$ is compact.*

Proof. Let s_0 be as in Proposition 4.4, and $s < s_0$. Suppose on the contrary that $\mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ is not compact. Then there exists a sequence $\{(u_i, \bar{u}_i)\} \subset \mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ such that any subsequence does not converge in $\mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. On the other hand, from Proposition 4.4, $\{(u_i, \bar{u}_i)\}$ has a subsequence converges to a broken gradient/continuation trajectory $((v_1, \bar{v}_1), \ldots, (v_N, \bar{v}_N))$ without bubble tree. In this case, N turns out to be 1 and $(v_1, \bar{v}_1) \in \mathcal{M}^0_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for generic $\{J_t\}_{t\in[0,1]}$ because of the virtual dimension counting. Since the subsequence preserves the condition $F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\delta})$, the limit $(v_1, \bar{v}_1)$ is in $\mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$, which contradicts that any subsequence does not converge in $\mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Thus $\mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ is compact. □

Theorem 4.6. *There exists s_0 such that, for $s < s_0$, $\mathcal{M}^{d,1}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\gamma} \in a_s$ has a suitable compactification whose boundary is given by*

$$\bigcup_{\boldsymbol{\theta} \in a_s} \hat{\mathcal{M}}^{d,0}_s(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\theta}, \boldsymbol{\delta}) \cup \bigcup_{\boldsymbol{\theta} \in c_S} \mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}^{d,0}_S(\boldsymbol{\theta}, \boldsymbol{\delta}).$$

Proof. The proof is based on the standard gluing-compsctness argument. For a pair $((u_1, \bar{u}_1), (u_2, \bar{u}_2)) \in \hat{\mathcal{M}}^{d,0}_s(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\theta}, \boldsymbol{\delta})$, the gluing procedure gives a unique connected component of $\mathcal{M}^1_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$, say $\mathcal{M}$, such that the pair is a compactifying point of $\mathcal{M}$. Since

$$\begin{aligned} F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\delta}) &= F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta}) + F_s(\boldsymbol{\theta}) - F_S(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\theta}) + f_s(\boldsymbol{\theta}) - f_S(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\delta}), \end{aligned}$$

$\mathcal{M}$ is contained in $\mathcal{M}^{d,1}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Similarly, we can glue the pairs of $\mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}^{d,0}_S(\boldsymbol{\theta}, \boldsymbol{\delta})$ to make connected components of $\mathcal{M}^{d,1}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

On the other hand, from Proposition 4.4, let $\{(u_i, \bar{u}_i)\} \subset \mathcal{M}^{d,1}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ be a sequence which converges to a broken gradient/continuation trajectory $((v_1, \bar{v}_1), \ldots, (v_N, \bar{v}_N))$ without bubble tree. In this case, N turns out to be 2 and $((v_1, \bar{v}_1), (v_2, \bar{v}_2)) \in \hat{\mathcal{M}}^0_s(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^0_{\rho_+}(\boldsymbol{\theta}, \boldsymbol{\delta})$ or $\mathcal{M}^0_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}^0_S(\boldsymbol{\theta}, \boldsymbol{\delta})$ for generic $\{J_t\}_{t\in[0,1]}$ because of the virtual dimension counting.

Suppose $((v_1, \bar{v}_1), (v_2, \bar{v}_2)) \in \hat{\mathcal{M}}^0_s(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^0_{\rho_+}(\boldsymbol{\theta}, \boldsymbol{\delta})$ is the limit of the sequence $\{(u_i, \bar{u}_i)\} \subset \mathcal{M}^{d,1}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. By Lemma 4.3, there exists s_0 such that, for $s < s_0$, $E(u_i) < \kappa$; since $0 < E(v_1)$, $0 < E(v_2)$ and $E(v_1) + E(v_2) \leq \limsup E(u_i)$,

$$E(v_1) = F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta}) < \kappa.$$

Then, by Theorem 3.8, there exists s_1 such that, for $s < s_1$, we have $\boldsymbol{\theta} \in a_s$. Moreover, since $\boldsymbol{\gamma}, \boldsymbol{\theta} \in a_s$, there exists s_2 such that, for $s < s_2$, $F_s(\boldsymbol{\gamma}) = f_s(\boldsymbol{\gamma})$ and $F_s(\boldsymbol{\theta}) = f_s(\boldsymbol{\theta})$; and $F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\theta})$. Since $(u_i, \bar{u}_i)$ is distinguished, i.e. $F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\delta})$,

$$\begin{aligned} F_s(\boldsymbol{\theta}) - F_S(\boldsymbol{\delta}) &= F_s(\boldsymbol{\theta}) - F_s(\boldsymbol{\gamma}) + F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\theta}) - f_s(\boldsymbol{\gamma}) + f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\theta}) - f_S(\boldsymbol{\delta}). \end{aligned}$$

Thus $((v_1, \bar{v}_1), (v_2, \bar{v}_2)) \in \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}_{\rho_+}^{d,0}(\boldsymbol{\theta}, \boldsymbol{\delta})$.

Suppose $((v_1, \bar{v}_1), (v_2, \bar{v}_2)) \in \mathcal{M}_{\rho_+}^0(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}_S^0(\boldsymbol{\theta}, \boldsymbol{\delta})$ is the limit of the sequence $\{(u_i, \bar{u}_i)\} \subset \mathcal{M}_{\rho_+}^{d,1}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Note that there exists s_0 such that, for $s < s_0$, $\varepsilon(s) < b_+ - a_+$ and $|f_s(\boldsymbol{\gamma})| < \varepsilon(s)$ for $\boldsymbol{\gamma} \in a_s$. Then, by Lemma 4.2, for $s < s_0$,

$$\begin{aligned} F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\theta}) &= E(v_1) + \int_{-\infty}^{\infty} \frac{d\rho_+(\tau)}{d\tau} \int_0^1 H(t, v_1(\tau, t)) dt d\tau \\ &> (S - s) \int_0^1 \min_{x \in M} H(t, x) dt \\ &\geq -a_+ \\ &= b_+ - a_+ - b_- - \kappa \\ &> \varepsilon(s) - b_- - \kappa \\ &> f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\theta}) - \kappa. \end{aligned} \tag{3}$$

Since $(u_i, \bar{u}_i)$ is distinguished, i.e. $F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\delta})$, there exists $m \in \mathbb{Z}$ such that

$$F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\theta}) = f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\theta}) + m\kappa, \tag{4}$$

$$F_S(\boldsymbol{\theta}) - F_S(\boldsymbol{\delta}) = f_S(\boldsymbol{\theta}) - f_S(\boldsymbol{\delta}) - m\kappa. \tag{5}$$

From (3) and (4), we obtain $m \geq 0$; and from (5), since $F_S(\boldsymbol{\theta}) - F_S(\boldsymbol{\delta}) = E(v_2) > 0$ and $\kappa > f_S(\boldsymbol{\theta}) - f_S(\boldsymbol{\delta})$, we obtain $0 \geq m$. Thus $m = 0$, which implies $((v_1, \bar{v}_1), (v_2, \bar{v}_2)) \in \mathcal{M}_{\rho_+}^{d,0}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}_S^{d,0}(\boldsymbol{\theta}, \boldsymbol{\delta})$. We obtain the compactification of $\mathcal{M}_{\rho_+}^{d,1}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. □

Similarly, we can prove the following theorems:

Theorem 4.7. *There exists s_0 such that, for $s < s_0$, $\mathcal{M}_{\rho_-}^{d,0}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\delta} \in a_s$ is compact.*

Theorem 4.8. *There exists s_0 such that, for $s < s_0$, $\mathcal{M}_{\rho_-}^{d,1}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\delta} \in a_s$ has a suitable compactification whose boundary is given by*

$$\bigcup_{\boldsymbol{\theta} \in c_S} \hat{\mathcal{M}}_S^{d,0}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}_{\rho_-}^{d,0}(\boldsymbol{\theta}, \boldsymbol{\delta}) \cup \bigcup_{\boldsymbol{\theta} \in a_s} \mathcal{M}_{\rho_-}^{d,0}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}_s^{d,0}(\boldsymbol{\theta}, \boldsymbol{\delta}).$$

For $s < s_0$, Theorem 4.5 and 4.7 allow us to define linear maps $\Phi_+ : A_s \to C_S$ and $\Phi_- : C_S \to A_s$ by

$$\Phi_+(\boldsymbol{\gamma}) := \sum_{\boldsymbol{\delta} \in c_S} \sharp \mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\delta})\boldsymbol{\delta}$$

for $\boldsymbol{\gamma} \in a_s$, and

$$\Phi_-(\boldsymbol{\gamma}) := \sum_{\boldsymbol{\delta} \in a_s} \sharp \mathcal{M}^{d,0}_{\rho_-}(\boldsymbol{\gamma}, \boldsymbol{\delta})\boldsymbol{\delta}$$

for $\boldsymbol{\gamma} \in c_S$. We call Φ_+ and Φ_- *continuations.* Then Corollary 3.11 and Theorem 4.6 and 4.8 imply the following theorem:

Theorem 4.9. *For $s < s_0$, $\Phi_+ \circ \partial_s = \partial_S \circ \Phi_+$ and $\Phi_- \circ \partial_S = \partial_s \circ \Phi_-$.*

Thus, for $s < s_0$, Φ_+ and Φ_- induce homomorphisms $\Phi_{+*} : H(A_s, \partial_s) \to H(C_S, \partial_S)$ and $\Phi_{-*} : H(C_S, \partial_S) \to H(A_s, \partial_s)$, respectively.

5. Homotopies of continuations

Let $\boldsymbol{\rho} = \boldsymbol{\rho}(w, \tau) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a smooth function such that

- $\frac{\partial \boldsymbol{\rho}}{\partial \tau} \geq 0$ when $\tau < 0$, and $\frac{\partial \boldsymbol{\rho}}{\partial \tau} \leq 0$ when $\tau > 0$,
- $w \mapsto \boldsymbol{\rho}(w, 0)$ is a monotone map onto $[s, S]$,
- $\boldsymbol{\rho}(0, \tau) \equiv s$,
- for w large enough,

$$\boldsymbol{\rho}(w, \tau) = \begin{cases} \rho_+(\tau + w) & \tau \leq 0, \\ \rho_-(\tau - w) & \tau \geq 0. \end{cases}$$

Let $\rho_w(\tau)$ denote $\boldsymbol{\rho}(w, \tau)$.

Let $s \in T$. We define $\mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\gamma}, \boldsymbol{\delta} \in c_s$ by

$$\mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta}) := \{(w, (u, \bar{u})) : w \in [0, \infty) \text{ and } (u, \bar{u}) \in \mathcal{M}_{\rho_w}(\boldsymbol{\gamma}, \boldsymbol{\delta})\}.$$

Theorem 5.1. *For generic $\{J_t\}_{t \in [0,1]}$, $\mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ is a finite dimensional smooth manifold with boundary $\partial \mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta}) = \{(0, (u, \bar{u})) \in \mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})\}$.*

Let $\mathcal{M}^k_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ denote the k-dimensional component of $\mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Moreover, we define $\mathcal{M}^d_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ by

$$\mathcal{M}^d_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta}) := \left\{(w, (u, \bar{u})) : w \in [0, \infty) \text{ and } (u, \bar{u}) \in \mathcal{M}^d_{\rho_w}(\boldsymbol{\gamma}, \boldsymbol{\delta})\right\}.$$

Let $\mathcal{M}^{d,k}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ denote the k-dimensional component of $\mathcal{M}^d_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

Lemma 5.2. *There exists s_0 such that for $s < s_0$*

$$E(u) < \kappa$$

for $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ and $(u, \bar{u}) \in \mathcal{M}^d_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

Proof. By Lemma 4.2, we obtain

$$\begin{aligned} E(u) &= F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) \\ &\quad - \int_{-\infty}^{0} \frac{\partial \rho_w(\tau)}{\partial \tau} \int_0^1 H(t, u(\tau, t)) dt d\tau \\ &\quad - \int_{0}^{\infty} \frac{\partial \rho_w(\tau)}{\partial \tau} \int_0^1 H(t, u(\tau, t)) dt d\tau \\ &\leq F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) + (S - s)(a_+ - a_-). \end{aligned}$$

Note that there exists s_0 such that, for $s < s_0$, $2\varepsilon(s) < \kappa - \|H\|$ and $|f_s(\boldsymbol{\gamma})| < \varepsilon(s)$ for $\boldsymbol{\gamma} \in a_s$ (and $|f_s(\boldsymbol{\delta})| < \varepsilon(s)$ for $\boldsymbol{\delta} \in a_s$). Since $(u, \bar{u})$ is distinguished, for $s < s_0$,

$$\begin{aligned} E(u) &\leq F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) + (S - s)(a_+ - a_-) \\ &= f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta}) + (S - s)(a_+ - a_-) \\ &\leq 2\varepsilon(s) + (S - s)(a_+ - a_-) \\ &< \kappa. \end{aligned}$$

□

Proposition 5.3. *There exists s_0 such that, for $s < s_0$, $\{(w_i, (u_i, \bar{u}_i))\} \subset \mathcal{M}^d_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ has a subsequece which converges to a broken gradient/continuation trajectory without bubble tree in the sense of Floer–Gromov convergence.*

Proof. By Lemma 4.2, $E(u_i)$ is uniformly bounded, and the Floer–Gromov compactness theorem implies that $\{(w_i, (u_i, \bar{u}_i))\}$ has a subsequence which converges to a gradient/continuation trajectory of

(A) $\partial \mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta}) = \{(0, (u, \bar{u})) \in \mathcal{M}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})\}$ when $\lim w_i = 0$,

(B) $\hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\theta}_1) \times \cdots \times \hat{\mathcal{M}}_s(\boldsymbol{\theta}_{i-1}, \boldsymbol{\theta}_i) \times \mathcal{M}_{\rho_w}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{i+1}) \times \hat{\mathcal{M}}_s(\boldsymbol{\theta}_{i+1}, \boldsymbol{\theta}_{i+1}) \times \cdots \times \hat{\mathcal{M}}_s(\boldsymbol{\theta}_{N-1}, \boldsymbol{\delta})$ when $\lim w_i = w \in (0, \infty)$,

(C) $\hat{\mathcal{M}}_s(\boldsymbol{\gamma}, \boldsymbol{\theta}_1) \times \cdots \times \hat{\mathcal{M}}_s(\boldsymbol{\theta}_{i-1}, \boldsymbol{\theta}_i) \times \mathcal{M}_{\rho_+}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{i+1}) \times \hat{\mathcal{M}}_S(\theta_{i+1}, \boldsymbol{\theta}_{i+2}) \times \cdots \times \hat{\mathcal{M}}_S(\boldsymbol{\theta}_{j-1}, \boldsymbol{\theta}_j) \times \mathcal{M}_{\rho_-}(\boldsymbol{\theta}_j, \boldsymbol{\theta}_{j+1}) \times \hat{\mathcal{M}}_s(\boldsymbol{\theta}_{j+1}, \boldsymbol{\theta}_{j+2}) \times \cdots \times \hat{\mathcal{M}}_s(\boldsymbol{\theta}_{N-1}, \boldsymbol{\delta})$ when $\lim w_i = \infty$

with bubble trees; the tail components of the bubble trees are

(i) a pseudoholomorphic sphere $v : S^2 \to M$,

(ii) a pseudoholomorphic disc $v : D^2 \to M$ with $\bar{v} : \partial D^2 \to L$ such that $v|_{\partial D^2} = \iota \circ \bar{v}$,

(iii) a pseudoholomorphic disc $v : D^2 \to M$ of $(v, \bar{v}) \in K(\iota)$.

But, since our Lagrangian immersion is exact, the bubbles of (i) and (ii) can not occur. Moreover, by Lemma 5.2, there exists s_0 such that for $s < s_0$ the symplectic area of the bubble trees is less than or equal to κ ($< \sigma$), and the bubbles of (iii) can not occur. Hence there is no bubble tree and the subsequence converges to the broken gradient/continuation trajectory. □

To define our homotopy, we need the following compactness theorems:

Theorem 5.4. *There exists* s_0 *such that, for* $s < s_0$, $\mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ *for* $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ *is compact.*

Proof. Let s_0 be as in Proposition 5.3, and $s < s_0$. Suppose on the contrary that $\mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ is not compact. Then there exists a sequence $\{(w_i, (u_i, \bar{u}_i))\} \subset \mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ such that any subsequence does not converge in $\mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. On the other hand, from Proposition 5.3, $\{(w_i, (u_i, \bar{u}_i))\}$ has a subsequence which converges to a broken gradient/continuation trajectory $((w, (v_1, \bar{v}_1)), \ldots, (w, (v_N, \bar{v}_N)))$ without bubble tree. In this case, N turns out to be 1 and $(w, (v_1, \bar{v}_1)) \in \mathcal{M}^0_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ for generic $\{J_t\}_{t\in[0,1]}$ because of the virtual dimension counting. Since the subsequence preserves the condition $F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta})$, the limit $(w, (v_1, \bar{v}_1))$ is in $\mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$, which contradicts that any subsequence does not converge in $\mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Thus $\mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ is compact. □

Theorem 5.5. *There exists* s_0 *such that, for* $s < s_0$, $\mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ *for* $\boldsymbol{\gamma}, \boldsymbol{\delta} \in a_s$ *has a suitable compactification whose boundary consists of*

(A) $\partial \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta}) = \{(0, (u, \bar{u})) \in \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})\}$,
(B1) $\bigcup_{\boldsymbol{\theta}\in a_s} \hat{\mathcal{M}}^{d,0}_s(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\theta}, \boldsymbol{\delta})$,
(B2) $\bigcup_{\boldsymbol{\theta}\in a_s} \mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}^{d,0}_s(\boldsymbol{\theta}, \boldsymbol{\delta})$,
(C) $\bigcup_{\boldsymbol{\theta}\in c_S} \mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^{d,0}_{\rho_-}(\boldsymbol{\theta}, \boldsymbol{\delta})$.

Note that, if $\boldsymbol{\gamma} = \boldsymbol{\delta}$, then

$$\partial \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = \{(u, \bar{u}) : u(\tau, t) = \gamma(t) \text{ and } \bar{u}(i) = \bar{\gamma}(i) \text{ for } = 0, 1\},$$

and, if $\boldsymbol{\gamma} \neq \boldsymbol{\delta}$, then $\partial \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta}) = \emptyset$ since the virtual dimension equals -1.

Proof. First, $\mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ has the boundary $\partial \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$, which is (A). Otherwise, we compactify $\mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ by the standard gluing-compsctness argument. For a pair $((u_1, \bar{u}_1), (w, (u_2, \bar{u}_2))) \in \hat{\mathcal{M}}^{d,0}_s(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\theta}, \boldsymbol{\delta})$, which is (B1), there exists a unique connected component of $\mathcal{M}^1_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$, say $\mathcal{M}$, such that the pair is a compactifying point of $\mathcal{M}$. Since

$$\begin{aligned} F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) &= F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta}) + F_s(\boldsymbol{\theta}) - F_s(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\theta}) + f_s(\boldsymbol{\theta}) - f_s(\boldsymbol{\delta}) \\ &= f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta}), \end{aligned}$$

$\mathcal{M}$ is contained in $\mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. Similarly, we can glue the pairs of (B2) and (C) to make connected components of $\mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$.

On the other hand, from Proposition 5.3, let $\{(w_i, (u_i, \bar{u}_i))\} \subset \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$ be a sequence which converges to a broken gradient/continuation trajectory $((w, (v_1, \bar{v}_1)), \ldots, (w, (v_N, \bar{v}_N)))$ of (B) or (C) without bubble tree. In this case, N turns out to be 2 and $((w, (v_1, \bar{v}_1)), (w, (v_2, \bar{v}_2)))$ is in $\hat{\mathcal{M}}^0_s(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^0_{\boldsymbol{\rho}}(\boldsymbol{\theta}, \boldsymbol{\delta})$, $\mathcal{M}^0_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \hat{\mathcal{M}}^0_s(\boldsymbol{\theta}, \boldsymbol{\delta})$ or $\mathcal{M}^0_{\rho+}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^0_{\rho_-}(\boldsymbol{\theta}, \boldsymbol{\delta})$ for generic $\{J_t\}_{t\in[0,1]}$ because of the virtual dimension counting.

Suppose $((v_1,\bar{v}_1),(w,(v_2,\bar{v}_2))) \in \hat{\mathcal{M}}^0_s(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \mathcal{M}^0_{\boldsymbol{\rho}}(\boldsymbol{\theta},\boldsymbol{\delta})$ is the limit of the sequence $\{(w_i,(u_i,\bar{u}_i))\} \subset \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma},\boldsymbol{\delta})$. By Lemma 5.2, there exists s_0 such that, for $s < s_0$, $E(u_i) < \kappa$; since $0 < E(v_1)$, $0 < E(v_2)$ and $E(v_1)+E(v_2) \leq \limsup E(u_i) \leq \kappa$,

$$E(v_1) = F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\theta}) < \kappa.$$

Then, by Theorem 3.8, there exists s_1 such that, for $s < s_1$, we have $\boldsymbol{\theta} \in a_s$. Moreover, since $\boldsymbol{\gamma},\boldsymbol{\theta},\boldsymbol{\delta} \in a_s$, there exists s_2 such that, for $s < s_2$, $F_s(\boldsymbol{\gamma}) = f_s(\boldsymbol{\gamma})$, $F_s(\boldsymbol{\theta}) = f_s(\boldsymbol{\theta})$ and $F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\delta})$. Thus $((v_1,\bar{v}_1),(w,(v_2,\bar{v}_2))) \in \hat{\mathcal{M}}^{d,0}_s(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\theta},\boldsymbol{\delta})$, which is (B1).

Suppose $((w,(v_1,\bar{v}_1)),(v_2,\bar{v}_2)) \in \hat{\mathcal{M}}^0_{\boldsymbol{\rho}}(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \mathcal{M}^0_s(\boldsymbol{\theta},\boldsymbol{\delta})$ is the limit of the sequence $\{(w_i,(u_i,\bar{u}_i))\} \subset \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma},\boldsymbol{\delta})$. By Lemma 5.2, there exists s_0 such that, for $s < s_0$, $E(u_i) < \kappa$; since $0 < E(v_1)$, $0 < E(v_2)$ and $E(v_1)+E(v_2) \leq \limsup E(u_i) \leq \kappa$,

$$E(v_2) = F_s(\boldsymbol{\theta}) - F_s(\boldsymbol{\delta}) < \kappa.$$

Then, by Theorem 3.10, there exists s_1 such that, for $s < s_1$, we have $\boldsymbol{\theta} \in a_s$. Moreover, since $\boldsymbol{\gamma},\boldsymbol{\theta},\boldsymbol{\delta} \in a_s$, there exists s_2 such that, for $s < s_2$, $F_s(\boldsymbol{\gamma}) = f_s(\boldsymbol{\gamma})$, $F_s(\boldsymbol{\theta}) = f_s(\boldsymbol{\theta})$ and $F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\delta})$. Thus $((w,(v_1,\bar{v}_1)),(v_2,\bar{v}_2)) \in \hat{\mathcal{M}}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \mathcal{M}^{d,0}_s(\boldsymbol{\theta},\boldsymbol{\delta})$, which is (B2).

Finally, suppose $((v_1,\bar{v}_1),(v_2,\bar{v}_2)) \in \mathcal{M}^0_{\rho_+}(\boldsymbol{\gamma},\boldsymbol{\theta}) \times \mathcal{M}^0_{\rho_-}(\boldsymbol{\theta},\boldsymbol{\delta})$ is the limit of the sequence $\{(w_i,(u_i,\bar{u}_i))\} \subset \mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma},\boldsymbol{\delta})$. Note that there exists s_0 such that, for $s < s_0$, $\varepsilon(s) < \min\{b_+ - a_+, a_- - b_-\}$ and $|f_s(\boldsymbol{\gamma})| < \varepsilon(s)$ for $\boldsymbol{\gamma} \in a_s$. Then, by Lemma 4.2, for $s < s_0$,

$$\begin{aligned}
F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\theta}) &= E(v_1) + \int_{-\infty}^{\infty} \frac{\partial \rho_+(\tau)}{\partial \tau} \int_0^1 H(t, v_1(\tau,t))dtd\tau \\
&> (S-s)\int_0^1 \min_{x\in M} H(t,x)dt \\
&\geq -a_+ \\
&= b_+ - a_+ - b_- - \kappa \\
&> f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\theta}) - \kappa, \qquad (6)
\end{aligned}$$

and similarly,

$$\begin{aligned}
F_S(\boldsymbol{\theta}) - F_s(\boldsymbol{\delta}) &= E(v_2) + \int_{-\infty}^{\infty} \frac{\partial \rho_-(\tau)}{\partial \tau} \int_0^1 H(t, v_1(\tau,t))dtd\tau \\
&> -(S-s)\int_0^1 \max_{x\in M} H(t,x)dt \\
&\geq a_- \\
&= b_+ + a_- - b_- - \kappa \\
&> f_S(\boldsymbol{\theta}) - f_s(\boldsymbol{\delta}) - \kappa. \qquad (7)
\end{aligned}$$

Since $(u_i, \bar{u}_i)$ is distinguished, i.e. $F_s(\boldsymbol{\gamma}) - F_s(\boldsymbol{\delta}) = f_s(\boldsymbol{\gamma}) - f_s(\boldsymbol{\delta})$, there exists $m \in \mathbb{Z}$ such that

$$F_s(\boldsymbol{\gamma}) - F_S(\boldsymbol{\theta}) = f_s(\boldsymbol{\gamma}) - f_S(\boldsymbol{\theta}) + m\kappa, \quad (8)$$

$$F_S(\boldsymbol{\theta}) - F_s(\boldsymbol{\delta}) = f_S(\boldsymbol{\theta}) - f_s(\boldsymbol{\delta}) - m\kappa. \quad (9)$$

From (6) and (8), we obtain $m \geq 0$; and from (7) and (9) we obtain $0 \geq m$. Thus $m = 0$, which implies $((v_1, \bar{v}_1), (v_2, \bar{v}_2)) \in \mathcal{M}^{d,0}_{\rho_+}(\boldsymbol{\gamma}, \boldsymbol{\theta}) \times \mathcal{M}^{d,0}_{\rho_-}(\boldsymbol{\theta}, \boldsymbol{\delta})$, which is (C). We obtain the compactification of $\mathcal{M}^{d,1}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta})$. □

For $s < s_0$, Theorem 5.4 allows us to define a linear map $H_s : A_s \to A_s$ by

$$H_s(\boldsymbol{\gamma}) := \sum_{\boldsymbol{\delta} \in a_s} \sharp \mathcal{M}^{d,0}_{\boldsymbol{\rho}}(\boldsymbol{\gamma}, \boldsymbol{\delta}) \boldsymbol{\delta}$$

for $\boldsymbol{\gamma} \in a_s$. We call H_s a *homotopy of continuations*. Then Corollary 3.12 and Theorem 5.5 imply the following theorem:

Theorem 5.6. *For $s < s_0$,*

$$\mathrm{id} + H_s \circ \partial_s + \partial_s \circ H_s + \Phi_- \circ \Phi_+ = 0.$$

Corollary 5.7. *For $s < s_0$,*

$$\Phi_{-*} \circ \Phi_{+*} = \mathrm{id} : H(A_s, \partial_s) \to H(A_s, \partial_s),$$

and hence $\Phi_{+*} : H(A_s, \partial_s) \to H(C_S, \partial_S)$ is injective.

6. The proof of Theorem 1.1

Let (M, ω) be a closed symplectic manifold or a non-compact symplectic manifold with convex end, and $\iota : L \to M$ an exact Lagrangian immersion from a closed manifold L. Suppose the non-injective points of $\iota : L \to M$ are transverse. Let φ^H_1 be the time one map generated by X_H.

Note that, for $\gamma(t) := \varphi^H_t(\delta(t))$, $(\gamma, \bar{\gamma}) \in c_1$ if and only if

$$\delta(t) \equiv p \in \iota(L) \cap (\varphi^{sH}_1)^{-1}(\iota(L)).$$

Hence we can identify $(\gamma, \bar{\gamma}) \in c_1$ with $(x, x') \in L \times L$ such that $\iota(x) = ((\varphi^H_1)^{-1} \circ \iota)(x')$. Suppose $\iota : L \to M$ and $\varphi^H_1 \circ \iota : L \to M$ intersect transversely. Then we slightly perturb H, if necessary, and choose $S \in T$ so that the numbers of the elements of $\{(x, x') \in L \times L : \iota(x) = ((\varphi^H_1)^{-1} \circ \iota)(x')\}$ and c_S are equal. First, our distinguished Floer homology gives

$$\sharp c_S \geq \dim H(C_S, \partial_S).$$

Secondly, from Theorem 3.14, there exists s_0 such that for $s < s_0$ and $s \in T$

$$\dim H(A_s, \partial_s) = \sum_{k=0}^{\dim L} \dim H_k(L; \mathbb{Z}_2).$$

Suppose $\|H\| < \kappa$. Then, from Corollary 5.7, there exists s_0 such that, for $s < s_0$ and $s \in T$, $\Phi_{+*} : H(A_s, \partial_s) \to H(C_S, \partial_S)$ is injective. Thus

$$\dim H(C_S, \partial_S) \geq \dim H(A_s, \partial_s).$$

Therefore we obtain

$$\sharp\left\{(x,x')\in L\times L : \iota(x) = ((\varphi_1^H)^{-1}\circ\iota)(x')\right\} \geq \sum_{k=0}^{\dim L} \dim H_k(L;\mathbb{Z}_2).$$

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