MORSE HOMOLOGY OF MANIFOLDS WITH BOUNDARY REVISITED

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Abstract. This re-certifying paper describes the details of the Morse homology of manifolds with boundary, introduced in [1], in terms of handlebody decompositions. First we carefully observe Riemannian metrics and Morse functions on manifolds with boundary so that their gradient vector fields are tangent to the boundary; secondly we confirm the stable manifolds and the unstable manifolds of critical points, and rigorously construct handlebody decompositions; and finally we re-certify that our Morse homology of manifolds with boundary is isomorphic to the absolute singular homology through connecting homomorphisms.

1. Introduction

Inspired by H. Hofer [3] and symplectic field theory [2], the author introduced a variant of Floer homology in [1] for Lagrangian submanifolds with Legendrian cylindrical end in a symplectic manifold with concave end, which can be thought as an infinite dimensional version of the following Morse homology of manifolds with boundary.

Let M be an n-dimensional oriented compact manifold with boundary N. Denote by $N_1, N_2, \dots, N_m$ the connected components of N. We fix a collar neighborhood $N \times [0,1) \subset M$, and denote by r the standard coordinate on the $[0,1)$-factor. Then we consider a Riemannain metric g on $M \setminus N$ such that, for $i = 1, \dots, m$,

$$g|_{N_i \times (0,1)}(x,r) = r^2 g_{N_i}(x) + dr \otimes dr,$$

where g_{N_i} is a Riemannian metric on N_i, and we consider a Morse–Smale function f on $M \setminus N$ such that, for $i = 1, \dots, m$,

$$f|_{N_i \times (0,1)}(x,r) = r^2 f_{N_i}(x) + c_i,$$

where $f_{N_i} : N_i \to \mathbb{R}$ is a Morse–Smale function on N_i and $c_i \in \mathbb{R}$ is a constant.

Let $Cr_k(f)$ be the set of the critical points $p \in M \setminus N$ of f with Morse index k, and $Cr_k^+(f_{N_i})$ the set of the critical points $\gamma \in N_i$ of f_{N_i} with Morse index k and $f_{N_i}(\gamma) > 0$, and similarly let $Cr_k^-(f_{N_i})$ be the set of the critical points $\delta \in N_i$ of f_{N_i} with Morse index k and $f_{N_i}(\delta) < 0$. We

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put $Cr_k^+(f_N) := \bigcup_{i=1}^m Cr_k^+(f_{N_i})$ and $Cr_k^-(f_N) := \bigcup_{i=1}^m Cr_k^-(f_{N_i})$. Then we define our Morse complex. Let $CM_k(f)$ be a free $\mathbb{Z}$ module

$$CM_k(f) := \bigoplus_{p \in Cr_k(f)} \mathbb{Z}p \oplus \bigoplus_{\gamma \in Cr_k^+(f_N)} \mathbb{Z}\gamma,$$

and $\partial_k : CM_k(f) \to CM_{k-1}(f)$ a linear map, for $p \in Cr_k(f)$,

$$\partial_k p := \sum_{p' \in Cr_{k-1}(f)} \sharp\mathcal{M}(p,p')p' + \sum_{\gamma \in Cr_{k-1}^+(f_N)} \sharp\mathcal{M}(p,\gamma)\gamma,$$

and for $\gamma \in Cr_k^+(f_N)$,

$$\begin{aligned} \partial_k \gamma := & \sum_{p \in Cr_{k-1}(f)} \sum_{\delta \in Cr_{k-1}^-(f_N)} \sharp\mathcal{N}(\gamma,\delta)\sharp\mathcal{M}(\delta,p)p \\ & + \sum_{\gamma' \in Cr_{k-1}^+(f_N)} \sum_{\delta \in Cr_{k-1}^-(f_N)} \sharp\mathcal{N}(\gamma,\delta)\sharp\mathcal{M}(\delta,\gamma')\gamma' \\ & + \sum_{\gamma' \in Cr_{k-1}^+(f_N)} \sharp\mathcal{N}(\gamma,\gamma')\gamma'. \end{aligned}$$

We give the precise definition of ∂_k in Section 7. An important remark is that $\delta \in Cr_k^-(f_N)$ is not a generator of $CM_k(f)$. Then our main theorem is that:

Theorem 1.1. *$(CM_*(f), \partial_*)$ is a chain complex, i.e. $\partial_{k-1} \circ \partial_k = 0$, and the homology is isomorphic to the absolute singular homology of M.*

As a corollary we obtain the following Morse type inequalities.

Corollary 1.2.

$$\sharp Cr_k(f) + \sharp Cr_k^+(f_N) \geq \dim H_k(M;\mathbb{R}).$$

There are several remarks on other related Morse homology. In [4], motivated by Seiberg–Witten Floer homology, Kronheimer–Mrowka also observed Morse homology of manifolds with boundary; they considered the double of a manifold with boundary and involution invariant Morse functions. In [5] F. Laudenbach also studied Morse homology of manifolds with boundary; his gradient vector field are also tangent to the boundary and his Morse complex counts trajectories of pseudo-gradient vector fields.

This paper consists of the following sections: first, in Section 2 we carefully observe Riemannian metrics and Morse functions on manifolds with boundary; in Section 3 we confirm the stable manifolds and the unstable manifolds of critical points, and fix their orientations; then, in Section 4 we rigorously construct handlebody decompositions, and in Section 5 we introduce relative cycles of critical points, which is a new technique and very important for the future applications; moreover, in Section 6 we prepare moduli spaces of gradient trajectories; and finally, in Section 7 we recall our Morse complex of manifolds with boundary, introduced in [1], and re-certify that our Morse homology is isomorphic to the absolute singular homology through connecting homomorphisms. Although it is important for Floer theory to consider compactifications of the moduli spaces of gradient trajectories, we do not mention them in this paper; the reader may refer to [1].

2. Riemannian metrics and Morse functions

In this section, we carefully observe Riemannian metrics and Morse functions on manifolds with boundary so that their gradient vector fields are tangent to the boundary.

Let M be an n-dimensional oriented compact manifold with boundary N. We denote by $N_1, N_2, \dots, N_m$ the connected components of N. Fix a collar neighborhood $N \times [0,1) \subset M$, and denote by r the standard coordinate on the $[0,1)$-factor.

Let g be a Riemannian metric on $M \setminus N$, and f a smooth function on $M \setminus N$. Just for simplicity, we consider g whose restriction on the collar neighborhood is

$$g|_{N_i \times (0,1)} = a g_{N_i} + dr \otimes dr,$$

where $a : (0,1) \to \mathbb{R}$ is a smooth function and g_{N_i} is a Riemannian metric on N_i. On the other hand, for the gluing analysis of gradient trajectories in Morse homology, we require the gradient vector field X_f of f with respect to g to be the following form on the collar neighborhood under the coordinate change of $r \in (0,1)$ and $t \in (-\infty, 0)$ by $r = e^t$:

$$X_f|_{N_i \times (-\infty,0)} = X_{f_{N_i}} + h_{N_i} \frac{\partial}{\partial t},$$

where f_{N_i} and h_{N_i} are smooth functions on N_i, and $X_{f_{N_i}}$ is the gradient vector field of f_{N_i} with respect to g_{N_i} on N_i. Note that

$$X_{f_{N_i}} + h_{N_i} \frac{\partial}{\partial t} = X_{f_{N_i}} + h_{N_i} r \frac{\partial}{\partial r}.$$

Lemma 2.1. *Suppose that N is connected. Let g be a Riemannian metric on $N \times (0,1)$ such that*

$$g = a g_N + dr \otimes dr,$$

where $a : (0,1) \to \mathbb{R}$ is a smooth function and g_N is a Riemannain metric on N, and let $f : N \times (0,1) \to \mathbb{R}$ be a smooth function whose gradient vector field X_f with respect to g is

$$X_f = X_{f_N} + h_N r \frac{\partial}{\partial r},$$

where f_N and h_N are non-constant smooth functions on N, and X_{f_N} is the gradient vector field of f_N with respect to g_N on N. Then

$$\begin{aligned} g &= (Ar^2 + B) g_N + dr \otimes dr, \\ f &= (Ar^2 + B) f_N + Cr^2 + D, \end{aligned}$$

where $A \neq 0, B, C, D \in \mathbb{R}$ are constants.

Proof. We write $f = f(x,r)$, $f_N = f_N(x)$, $h_N = h_N(x)$ and $a = a(r)$ for $(x,r) \in N \times (0,1)$, and we denote by d_N the exterior derivative on N. From g and X_f,

$$g(X_f, \cdot) = a(r) d_N f_N(x) + h_N(x) r dr$$

On the other hand,

$$df(x,r) = d_N f(x,r) + \frac{\partial f(x,r)}{\partial r} dr.$$

So we have

$$d_N f(x,r) = d_N\{a(r) f_N(x)\}, \quad (1)$$

$$\frac{\partial f(x,r)}{\partial r} = h_N(x) r. \quad (2)$$

Since N is connected, and from (1),

$$f(x,r) = a(r) f_N(x) + c(r),$$

where $c = c(r)$ is a smooth function on $(0,1)$, and then

$$\frac{\partial f(x,r)}{\partial r} = \frac{da(r)}{dr} f_N(x) + \frac{dc(r)}{dr}. \quad (3)$$

From (2) and (3),

$$h_N(x) = \frac{1}{r}\frac{da(r)}{dr} f_N(x) + \frac{1}{r}\frac{dc(r)}{dr}.$$

Since we assume that $f_N(x)$ and $h_N(x)$ are non-constant smooth functions,

$$\begin{aligned} \frac{1}{r}\frac{da(r)}{dr} &= 2A, \\ \frac{1}{r}\frac{dc(r)}{dr} &= 2C, \end{aligned}$$

where $A \neq 0$ and $C \in \mathbb{R}$ are constants, and hence

$$\begin{aligned} a(r) &= Ar^2 + B, \\ c(r) &= Cr^2 + D, \end{aligned}$$

where $B, D \in \mathbb{R}$ are constants. Then we obtain

$$\begin{aligned} g(x,r) &= (Ar^2 + B) g_N(x) + dr \otimes dr, \\ f(x,r) &= (Ar^2 + B) f_N(x) + Cr^2 + D. \end{aligned}$$

□

Corollary 2.2.

$$X_f = X_{f_N} + 2(Af_N + C) r \frac{\partial}{\partial r}.$$

We call a Riemannian metric g on $M \setminus N$ *cone end* if g satisfies

$$g|_{N_i\times(0,1)} = r^2 g_{N_i} + dr\otimes dr,$$

where g_{N_i} is a Riemannian metric on N_i, and we call a Morse function f on $M \setminus N$ *cone end* if f satisfies

$$f|_{N_i\times(0,1)} = r^2 f_{N_i} + c_i,$$

where f_{N_i} is a Morse function on N_i and $c_i \in \mathbb{R}$ is a constant. On the other hand, in [1] we used Riemannian metrics g and Morse functions f on $M\setminus N$ such that

$$\begin{aligned} g|_{N_i\times(0,1)} &= r g_{N_i} + r^{-1} dr\otimes dr, \\ f|_{N_i\times(0,1)} &= r f_{N_i} + c_i, \end{aligned}$$

which we called *horn end*. The following lemma implies that there is no essential difference between cone end and horn end for our purpose; the case of $\bar{a} = 0$ is cone end, and $a = -1$ is horn end.

Lemma 2.3. *For $a + 2 \neq 0$ and $\bar{a} + 2 \neq 0$, let $\bar{r}^{\bar{a}+2} = \left(\frac{\bar{a}+2}{a+2}\right)^2 r^{a+2}$, $\bar{g}_N = \left(\frac{a+2}{\bar{a}+2}\right)^2 g_N$ and $\bar{f}_N = \left(\frac{a+2}{\bar{a}+2}\right)^2 f_N$. Then*

$$\begin{aligned} \bar{r}^{\bar{a}+2}\bar{g}_N + \bar{r}^{\bar{a}} d\bar{r}\otimes d\bar{r} &= r^{a+2} g_N + r^a dr\otimes dr, \\ \bar{r}^{\bar{a}+2}\bar{f}_N + c &= r^{a+2} f_N + c. \end{aligned}$$

Proof. Direct computations. □

Moreover, instead of cone end or horn end, we can also use Riemannian metrics g and Morse functions f on $M \setminus N$, or M, which satisfy

$$\begin{aligned} g|_{N_i\times(0,1)} &= (r^2+1) g_{N_i} + dr\otimes dr, \\ f|_{N_i\times(0,1)} &= (r^2+1) f_{N_i} + c_i \end{aligned}$$

since their gradient vector field on $N_i \times (0, 1)$ is completely the same as the one of cone end:

$$X_f|_{N_i\times(0,1)} = X_{f_{N_i}} + 2 f_{N_i} r \frac{\partial}{\partial r}.$$

We call such Riemannian metrics and Morse functions *doubling end*.

We remark that, for all the types of pairs of a Riemannian metric and a Morse function above, we can define their Morse complex in the same way.

In this paper we use cone end.

Lemma 2.4. *Let f be a cone end Morse function. If $\gamma \in N_i$ is a critical point of f_{N_i}, then $f_{N_i}(\gamma) \neq 0$.*

Proof. Since $df|_{N_i\times(0,1)} = r^2 df_{N_i} + 2r f_{N_i} dr$ and the critical points of a Morse function are isolated, $f_{N_i}(\gamma) \neq 0$. □

Hence we can divide the critical points x of f_{N_i} into two groups; one is $f_{N_i}(x) > 0$, and the other is $f_{N_i}(x) < 0$.

Moreover we have the following lemma:

Lemma 2.5. *Let g and f be cone end. Then there is no map $u : \mathbb{R} \to M \setminus N$ which satisfies $du/dt = -X_f \circ u$ with $\lim_{t\to-\infty} u(t) \in N_i \times \{0\}$ and $\lim_{t\to\infty} u(t) \in N_i \times \{0\}$.*

Proof. Suppose that a map $u : \mathbb{R} \to M \setminus N$ satisfies $du/dt = -X_f \circ u$ with $\lim_{t\to-\infty} u(t) \in N_i\times\{0\}$ and $\lim_{t\to\infty} u(t) \in N_i\times\{0\}$. Since $du/dt = -X_f\circ u$, $\lim_{t\to-\infty} f(u(t)) > \lim_{t\to\infty} f(u(t))$, which contradicts $\lim_{t\to-\infty} f(u(t)) = \lim_{t\to\infty} f(u(t)) = c_i$. □

On the other hand, there may exist

- a non-constant map $u : \mathbb{R} \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$, and
- a map $u : \mathbb{R} :\to M \setminus N$ which satisfies $du/dt = -X_f \circ u$ with $\lim_{t\to-\infty} u(t) \in N_i \times \{0\}$ and $\lim_{t\to\infty} u(t) \in N_j \times \{0\}$ if $c_i > c_j$; this was pointed out by T. Nishino and Y. Nohara.

We remark that, although their proofs need slight modifications, Lemma 2.4 and Lemma 2.5 also hold for horn end and doubling end.

3. Stable manifolds and unstable manifolds

First we prepare notation. Let M be an n-dimensional oriented compact manifold with boundary N as before, and g and f a cone end Riemannian metric and a cone end Morse function on $M \setminus N$, respectively. We fix an orientation of N so that, for an oriented basis $\{v_1, \dots, v_{n-1}\}$ of T_pN and an outward-pointing vector $v_{out} \in T_pM$, the orientations of $\{v_{out}, v_1, \dots, v_{n-1}\}$ and T_pM coincide.

We define $Cr_k(f)$ to be the set of the critical points $p \in M \setminus N$ of f with Morse index k, and $Cr^+_k(f_{N_i})$ the set of the critical points $\gamma \in N_i$ of f_{N_i} with Morse index k and $f_{N_i}(\gamma) > 0$, and similarly we define $Cr^-_k(f_{N_i})$ to be the set of the critical points $\delta \in N_i$ of f_{N_i} with Morse index k and $f_{N_i}(\delta) < 0$. We put $Cr^+_k(f_N) := \bigcup_{i=1}^m Cr^+_k(f_{N_i})$ and $Cr^-_k(f_N) := \bigcup_{i=1}^m Cr^-_k(f_{N_i})$.

Let X_f be the gradient vector field on $M\setminus N$ of a cone end Morse function f with respect to a cone end Riemannian metric g. Then the restriction of X_f on the collar neighborhood is

$$X_f|_{N_i\times(0,1)} = X_{f_{N_i}} + 2f_{N_i} r\frac{\partial}{\partial r}.$$

Hence we define a vector field $\overline{X}_f$ on M by

$$\overline{X}_f := \begin{cases} X_f, & \text{on } M \setminus N, \\ X_{f_{N_i}}, & \text{on } N_i \times \{0\}, \end{cases}$$

and denote by $\overline{\varphi}_t : M \to M$ the isotopy of $-\overline{X}_f$, i.e. $\overline{\varphi}_t$ is given by $d\overline{\varphi}_t/dt = -\overline{X}_f \circ \overline{\varphi}_t$ and $\overline{\varphi}_0(x) = x$.

Let $B^k := \{(x_1, \dots, x_k) : x_1^2 + \cdots + x_k^2 < 1\}$ be the k-dimensional open ball, and $\partial B^k := \overline{B^k} \setminus B^k$. Moreover, we define the k-dimensional open half-ball $H^k := \{(x_1, \dots, x_k) : x_1^2 + \cdots + x_k^2 < 1, x_k \geq 0\}$ and $\partial H^k := \{(x_1, \dots, x_k) \in H^k : x_k = 0\}$.

Now we define stable manifolds and unstable manifolds for critical points. For $p \in Cr_k(f)$, we define the stable manifold S_p of p by

$$S_p := \left\{ x \in M : \lim_{t \to +\infty} \overline{\varphi}_t(x) = p \right\},$$

and the unstable manifold U_p of p by

$$U_p := \left\{ x \in M : \lim_{t \to -\infty} \overline{\varphi}_t(x) = p \right\}.$$

Since $\overline{X}_f$ is tangent to N, S_p and U_p are contained in $M \setminus N$. Note that S_p is diffeomorphic to B^{n-k}, and U_p is diffeomorphic to B^k. Moreover, S_p and U_p intersect transversely at p. We fix orientations of S_p and U_p so that the orientations of $T_pS_p \oplus T_pU_p$ and T_pM coincide.

Next, for $\gamma \in Cr_k^+(f_{N_i})$, we define the stable manifold S_γ of γ by

$$S_\gamma := \left\{ x \in M : \lim_{t \to +\infty} \overline{\varphi}_t(x) = (\gamma, 0) \in N_i \times \{0\} \right\},$$

and the unstable manifold U_γ of γ by

$$U_\gamma := \left\{ x \in M : \lim_{t \to -\infty} \overline{\varphi}_t(x) = (\gamma, 0) \in N_i \times \{0\} \right\}.$$

Since $f_{N_i}(\gamma) > 0$, S_γ is contained in M, and U_γ is contained in $N_i \times \{0\} \subset M$. Note that U_γ is diffeomorphic to B^k, S_γ is diffeomorphic to H^{n-k}, and $S_\gamma \cap (N_i \times \{0\})$ is diffeomorphic to $\partial H^{n-k} \cong B^{n-1-k}$. Moreover S_γ and U_γ intersect transversely at $(\gamma, 0) \in N_i \times \{0\}$.

We fix orientations of S_γ and U_γ so that the orientations of $T_\gamma S_\gamma \oplus T_\gamma U_\gamma$ and $T_\gamma M$ coincide. Moreover, we fix an orientation of $S_\gamma \cap (N_i \times \{0\})$ so that, for an oriented basis $\{v_1, \dots, v_{n-1-k}\}$ of $T_\gamma(S_\gamma \cap (N_i \times \{0\}))$ and an outward-pointing vector $v_{out} \in T_\gamma M$, the orientations of $\{v_{out}, v_1, \dots, v_{n-1-k}\}$ and $T_\gamma S_\gamma$ coincide. Then the orientations of $T_\gamma(S_\gamma \cap (N_i \times \{0\})) \oplus T_\gamma U_\gamma$ and $T_\gamma N_i$ coincide.

Similarly, for $\delta \in Cr_k^-(f_{N_i})$, we define the stable manifold S_δ of δ by

$$S_\delta := \left\{ x \in M : \lim_{t \to +\infty} \overline{\varphi}_t(x) = (\delta, 0) \in N_i \times \{0\} \right\},$$

and the unstable manifold U_δ of δ by

$$U_\delta := \left\{ x \in M : \lim_{t \to -\infty} \overline{\varphi}_t(x) = (\delta, 0) \in N_i \times \{0\} \right\}.$$

Since $f_{N_i}(\delta) < 0$, S_δ is contained in $N_i \times \{0\} \subset M$, and U_δ is contained in M. Note that S_δ is diffeomorphic to B^{n-1-k}, U_δ is diffeomorphic to H^{k+1}, and $U_\delta \cap (N_i \times \{0\})$ is diffeomorphic to $\partial H^{k+1} \cong B^k$. Moreover, S_δ and U_δ intersect transversely at $(\delta, 0) \in N_i \times \{0\}$.

We fix orientations of S_δ and U_δ so that the orientations of $T_\delta S_\delta \oplus T_\delta U_\delta$ and $T_\delta M$ coincide. Moreover, we fix an orientation of $U_\delta \cap (N_i \times \{0\})$ so that, for an oriented basis $\{v_{n-k}, \dots, v_n\}$ of $T_\delta(U_\delta \cap (N_i \times \{0\}))$ and an outward-pointing vector $v_{out} \in T_\delta M$, the orientations of $\{v_{out}, v_{n-k}, \dots, v_n\}$ and $T_\delta U_\delta$ coincide. Then the difference of the orientations of $T_\delta S_\delta \oplus T_\delta(U_\delta \cap (N_i \times \{0\}))$ and $T_\delta N_i$ is $(-1)^{n-k-1}$.

4. **Handlebody decompositions**

Let M be an n-dimensional oriented compact manifold with boundary N as before, and g and f a cone end Riemannian metric and a cone end Morse function on $M \setminus N$, respectively. Moreover we assume that f satisfies the Morse–Smale conditions in the following sense:

- for $p, p' \in \bigcup_{k=0}^{n} Cr_k(f)$, U_p and $S_{p'}$ intersect transversely in $M \setminus N$,
- for $\theta, \theta' \in \bigcup_{k=0}^{n-1} Cr_k^+(f_{N_i}) \cup \bigcup_{k=0}^{n-1} Cr_k^-(f_{N_i})$, U_θ and $S_{\theta'}$ intersect transversely in N_i,
- for $p \in \bigcup_{k=0}^{n} Cr_k(f)$ and $\gamma \in \bigcup_{k=0}^{n-1} Cr_k^+(f_N)$, U_p and S_γ intersect transversely in $M \setminus N$,
- for $\delta \in \bigcup_{k=0}^{n-1} Cr_k^-(f_N)$ and $p \in \bigcup_{k=0}^{n} Cr_k(f)$, U_δ and S_p intersect transversely in $M \setminus N$, and
- for $\delta \in \bigcup_{k=0}^{n-1} Cr_k^-(f_{N_i})$ and $\gamma \in \bigcup_{k=0}^{n-1} Cr_k^+(f_{N_j})$ with $c_i > c_j$, U_δ and S_γ intersect transversely in $M \setminus N$.

In fact we can prove that generic cone end Morse functions satisfy the above Morse–Smale conditions by the standard generosity arguments.

Recall that $\overline{X}_f$ is the vector field on M defined by

$$\overline{X}_f := \begin{cases} X_f, & \text{on } M \setminus N, \\ X_{f_{N_i}}, & \text{on } N_i \times \{0\}. \end{cases}$$

We call a map $u : \mathbb{R} \to M$ a gradient trajectory from x to y if $du/dt = -\overline{X}_f \circ u$ with $\lim_{t\to-\infty} u(t) = x$ and $\lim_{t\to\infty} u(t) = y$. Then we can prove the following lemma:

Lemma 4.1. *Let f be a cone end Morse–Smale function on $M \setminus N$. For $p \in Cr_k(f)$ and $p' \in Cr_l(f)$, there is no non-constant gradient trajectory from p to p' if $k \le l$.*

Proof. Let $u : \mathbb{R} \to M$ be a non-constant gradient trajectory from p to p'. Then the image of u is contained in $U_p \cap S_{p'}$. Since U_p and $S_{p'}$ intersect transversely in $M \setminus N$ so that $\dim U_p \cap S_{p'} = k - l$, and since the dimension of the image of u is 1, there is no such gradient trajectory if $k - l \le 0$. □

Similarly we can prove the following lemma. We omit the proof:

Lemma 4.2. *Let f be a cone end Morse–Smale function on M.*
(1) For $\theta \in Cr_k^+(f_{N_i}) \cup Cr_k^-(f_{N_i})$ and $\theta' \in Cr_l^+(f_{N_i}) \cup Cr_l^-(f_{N_i})$, there is no non-constant gradient trajectory from θ to θ' if $k \le l$.
(2) For $p \in Cr_k(f)$ and $\gamma \in Cr_l^+(f_N)$, there is no non-constant gradient

trajectory from p to γ if $k \le l$.
(3) For $\delta \in Cr_k^-(f_N)$ and $p \in Cr_l(f)$, there is no non-constant gradient trajectory from δ to p if $k+1 \le l$.
(4) For $\delta \in Cr_k^-(f_{N_i})$ and $\gamma \in Cr_l^+(f_{N_j})$ with $i \neq j$, there is no non-constant gradient trajectory from δ to γ if $k+1 \le l$.

Now we construct a handlebody decomposition of M:

Theorem 4.3. *Let f be a cone end Morse–Smale function on $M \setminus N$. Then there exists a sequence of open subsets $M^{-1} = \tilde{M}^0 = \emptyset \subset M^0 \subset \tilde{M}^1 \subset M^1 \subset \cdots \subset \tilde{M}^n \subset M^n = M$ such that*

- *$\partial\tilde{M}^k := \overline{\tilde{M}^k} \setminus \tilde{M}^k$ and $\partial M^k := \overline{M^k} \setminus M^k$ are smooth and transversal to $\overline{X}_f$, where $\overline{\tilde{M}^k}$ and $\overline{M^k}$ are the closures of $\tilde{M}^k$ and M^k in M, respectively,*
- *for $\delta \in Cr_{k-1}^-(f_N)$, ∂M^{k-1} and U_δ intersect transversely, and $U_\delta \setminus M^{k-1}$ is diffeomorphic to the k-dimensional closed half-ball,*
- *$\overline{M^{k-1}} \cup \bigcup_{\delta \in Cr_{k-1}^-(f_N)} U_\delta$ is a deformation retract of $\overline{\tilde{M}^k}$;*
- *for $p \in Cr_k(f)$, $\partial\tilde{M}^k$ and U_p intersect transversely, and $U_p \setminus \tilde{M}^k$ is diffeomorphic to the k-dimensional closed ball,*
- *for $\gamma \in Cr_k^+(f_N)$, $\partial\tilde{M}^k$ and U_γ intersect transversely, and $U_\gamma \setminus \tilde{M}^k$ is diffeomorphic to the k-dimensional closed ball, and*
- *$\overline{\tilde{M}^k} \cup \bigcup_{p \in Cr_k(f)} U_p \cup \bigcup_{\gamma \in Cr_k^+(f_N)} U_\gamma$ is a deformation retract of $\overline{M^k}$.*

We call the sequence $M^{-1} = \emptyset \subset M^0 \subset M^1 \subset \cdots \subset M^n = M$ a handlebody decomposition of M.

Proof. We construct $\tilde{M}^k$ and M^k inductively. For $p \in Cr_0(f)$, let $(x_1, \ldots, x_n)$ be a local coordinate centered at p in M and $B_\varepsilon(p) := \{x_1^2 + \cdots + x_n^2 < \varepsilon^2\} \subset M$; and for $\gamma \in Cr_0^+(f_N)$, let $(y_1, \ldots, y_{n-1})$ be a local coordinate centered at γ in N and $B_\varepsilon(\gamma) := \{y_1^2 + \cdots + y_{n-1}^2 + r^2 < \varepsilon^2\} \subset M$. Here we put $\partial B_\varepsilon(\gamma) := \{y_1^2 + \cdots + y_{n-1}^2 + r^2 = \varepsilon^2, r \ge 0\}$. Then we may take ε to be small so that the closures of $B_\varepsilon(p)$ and $B_\varepsilon(\gamma)$ in M are mutually disjoint, and $\partial B_\varepsilon(p)$ and $\partial B_\varepsilon(\gamma)$ are transversal to $\overline{X}_f$. We define $M^0 := \bigcup_{p \in Cr_0(f)} B_\varepsilon(p) \cup \bigcup_{\gamma \in Cr_0^+(f_N)} B_\varepsilon(\gamma)$, and then

- $\bigcup_{p \in Cr_0(f)} U_p \cup \bigcup_{\gamma \in Cr_0^+(f_N)} U_\gamma$ is a deformation retract of $\overline{M^0}$, and
- ∂M^0 is smooth and transversal to $\overline{X}_f$.

This is the first step of $k = 0$ to construct the handlebody decomposition of M. Suppose we have $M^{-1} = \tilde{M}^0 = \emptyset \subset M^0 \subset \tilde{M}^1 \subset M^1 \subset \cdots \subset \tilde{M}^{k-1} \subset M^{k-1}$ as in the theorem. Since ∂M^{k-1} is smooth and transversal to $\overline{X}_f$, for $\delta \in Cr_{k-1}^-(f_N)$,

- ∂M^{k-1} and U_δ intersect transversely,

and moreover, since f is Morse–Smale,

- $U_\delta \setminus M^{k-1}$ is diffeomorphic to the k-dimensional closed half-ball, where the k-dimensional closed half-ball is diffeomorphic to $\{y_1^2+\cdots+y_{k-1}^2+r^2 \leq 1, r \geq 0\}$. Hence we may attach half k-handles for $\delta \in Cr_{k-1}^-(f_N)$ to $\overline{M^{k-1}}$ and obtain $\tilde{M}^k$ so that

- $\overline{M^{k-1}} \cup \bigcup_{\delta \in Cr_{k-1}^-(f_N)} U_\delta$ is a deformation retract of $\overline{\tilde{M}^k}$, and
- $\partial\tilde{M}^k$ is smooth and transversal to $\overline{X}_f$,

where the half k-handle is diffeomorphic to $\{y_1^2+\cdots+y_{k-1}^2+r^2 \leq 1, r \geq 0\} \times B^{n-k}$ and the attaching map is from $\{y_1^2+\cdots+y_{k-1}^2+r^2 = 1, r \geq 0\} \times B^{n-k}$ to ∂M^{k-1}. Since $\partial\tilde{M}^k$ is transversal to $\overline{X}_f$,

- for $p \in Cr_k(f)$, $\partial\tilde{M}^k$ and U_p intersect transversely, and
- for $\gamma \in Cr_k^+(f_N)$, $\partial\tilde{M}^k$ and U_γ intersect transversely,

and moreover, since f is Morse–Smale,

- $U_p \setminus \tilde{M}^k$ is diffeomorphic to the k-dimensional closed ball, and
- $U_\gamma \setminus \tilde{M}^k$ is diffeomorphic to the k-dimensional closed ball.

Hence we may attach k-handles for $p \in Cr_k(f)$ and $\gamma \in Cr_k^+(f_N)$ to $\overline{\tilde{M}^k}$ and obtain M^k so that

- $\overline{\tilde{M}^k} \cup \bigcup_{p \in Cr_k(f)} U_p \cup \bigcup_{\gamma \in Cr_k^+(f_N)} U_\gamma$ is a deformation retract of $\overline{M^k}$, and
- ∂M^k is smooth and transversal to $\overline{X}_f$.

Then these $\tilde{M}^k$ and M^k satisfy the conditions as in the theorem. Therefore we obtain the handlebody decomposition $M^{-1} = \emptyset \subset M^0 \subset M^1 \subset \cdots \subset M^n = M$ by induction. □

5. Relative cycles

In this section we introduce relative cycles of critical points to define our Morse complex on manifolds with boundary. This new technique is also very important for the future applications.

First we confirm notation. Let M be an n-dimensional oriented compact manifold with boundary N as before, and g and f a cone end Riemannian metric and a cone end Morse–Smale function on $M \setminus N$, respectively. Moreover, let $M^{-1} = \tilde{M}^0 = \emptyset \subset M^0 \subset \tilde{M}^1 \subset M^1 \subset \cdots \subset \tilde{M}^n \subset M^n = M$ be the sequence of open subsets constructed in Theorem 4.3.

Let $B_r^k := \{(x_1, \ldots, x_k) : x_1^2 + \cdots + x_k^2 < r^2\}$ and $\partial B_r^k := \overline{B_r^k} \setminus B_r^k$, and similarly, let $H_r^k := \{(x_1, \ldots, x_k) : x_1^2 + \cdots + x_k^2 < r^2, x_k \geq 0\}$ and $\partial H_r^k := \{(x_1, \ldots, x_k) \in H_r^k : x_k = 0\}$.

For $0 < \varepsilon < 1$, we define a diffeomorphism $\rho_\varepsilon : M \to \rho_\varepsilon(M) \subset M$ so that $\rho_\varepsilon(x) = x$ for $x \notin [0, \varepsilon) \times N$, and $\rho_\varepsilon(x, r) = (x, r/2 + \varepsilon/2)$ for $(x, r) \in N \times [0, \varepsilon/2)$.

For $p \in Cr_k(f)$, we fix a diffeomorphism $\psi_p : B_1^k \to U_p$ with $\psi_p(0) = p$. Since $\partial\tilde{M}^k$ and U_p intersect transversely, and $U_p \setminus \tilde{M}^k$ is diffeomorphic to the k-dimensional closed ball, there exists $0 < r_p < 1$ such that $\psi_p(\partial B_{r_p}^k) \subset \tilde{M}^k$.

We call the restriction $\psi_p|_{\overline{B^k_{r_p}}} : \overline{B^k_{r_p}} \to U_p$ a relative cycle for p, and denote by $\sigma_p : \overline{B^k_{r_p}} \to U_p$. Note that σ_p is an embedding, and S_p and the image of σ_p intersect transversely and positively at p. Similarly, for $\gamma \in Cr^+_k(f_N)$, we fix a diffeomorphism $\psi_\gamma : B^k_1 \to U_\gamma$ with $\psi_\gamma(0) = \gamma$. Since

- $\overline{\tilde{M}^k} \cup \bigcup_{p \in Cr_k(f)} U_p \cup \bigcup_{\gamma \in Cr^+_k(f_N)} U_\gamma$ is a deformation retract of $\overline{M^k}$, and
- $U_\gamma \setminus \tilde{M}^k$ is diffeomorphic to the k-dimensional closed ball,

there exist $0 < \varepsilon < 1$ and $0 < r_\gamma < 1$ such that $\rho_\varepsilon \circ \psi_\gamma(B^k_{r_\gamma}) \subset M^k$ and $\rho_\varepsilon \circ \psi_\gamma(\partial B^k_{r_\gamma}) \subset \tilde{M}^k$. Note that, since $\partial \tilde{M}^k$ is smooth and transversal to $\overline{X}_f$, $\varphi_t(\rho_\varepsilon \circ \psi_\gamma(\partial B^k_{r_\gamma}))$ is contained in $\tilde{M}^k$, for $t \geq 0$, where φ_t is the isotopy of $-X_f$. Moreover, since $\varepsilon > 0$ and

- $\overline{M^{k-1}} \cup \bigcup_{\delta \in Cr^-_{k-1}(f_N)} U_\delta$ is a deformation retract of $\overline{\tilde{M}^k}$, and
- ∂M^{k-1} are smooth and transversal to $\overline{X}_f$,

there exists $T_\gamma > 0$ such that $\varphi_{T_\gamma}(\rho_\varepsilon \circ \psi_\gamma(\partial B^k_{r_\gamma})) \subset M^{k-1}$. Now we define a map $\sigma_\gamma : \overline{B^k_{r_\gamma}} \cup (\partial B^k_{r_\gamma} \times [0, T_\gamma]) \to M$ as follows: first we glue $\overline{B^k_{r_\gamma}}$ and $\partial B^k_{r_\gamma} \times [0, T_\gamma]$ by the natural identification of $\partial B^k_{r_\gamma} \subset \overline{B^k_{r_\gamma}}$ with $\partial B^k_{r_\gamma} \times \{0\} \subset \partial B^k_{r_\gamma} \times [0, T_\gamma]$; and then we define σ_γ by $\sigma_\gamma(x) := \rho_\varepsilon \circ \psi_\gamma(x)$ if $x \in \overline{B^k_{r_\gamma}}$, and $\sigma_\gamma(x, t) := \varphi_t(\rho_\varepsilon \circ \psi_\gamma(x))$ if $(x, t) \in \partial B^k_{r_\gamma} \times [0, T_\gamma]$. We call $\sigma_\gamma : \overline{B^k_{r_\gamma}} \cup (\partial B^k_{r_\gamma} \times [0, T_\gamma]) \to M$ a relative cycle of γ. Note that σ_γ is a piecewise embedding, and S_γ and the image of σ_γ intersect transversely and positively at $\rho_\varepsilon(\gamma)$.

6. **Gradient trajectories**

Let M be an n-dimensional oriented compact manifold with boundary N as before, and g and f a cone end Riemannian metric and a cone end Morse–Smale function on $M \setminus N$, respectively.

Let $p, p' \in M \setminus N$ be critical points of f, and $u : \mathbb{R} \to M \setminus N$ a map which satisfies $du/dt = -X_f \circ u$ with $\lim_{t \to -\infty} u(t) = p$ and $\lim_{t \to \infty} u(t) = p'$. We call such u a gradient trajectory from p to p', and moreover, we call such u up to parameter shift an unparameterized gradient trajectory. We define $\mathcal{M}(p, p')$ to be the set of the unparameterized gradient trajectories from p to p'. Since an intersection point $x \in S_{p'} \cap \sigma_p(\partial B^k_{r_p})$ corresponds to the unparameterized gradient trajectory from p to p' through x, $\mathcal{M}(p, p')$ can be identified with $S_{p'} \cap \sigma_p(\partial B^k_{r_p})$. Let $p \in Cr_k(f)$ and $p' \in Cr_{k-1}(f)$. For $x \in S_{p'} \cap \sigma_p(\partial B^k_{r_p})$, we define $\epsilon_x := 1$ if the orientations of $T_x S_{p'} \oplus T_x \sigma_p(\partial B^k_{r_p})$ and $T_x M$ coincide, and $\epsilon_x := -1$ otherwise. Then we assign ϵ_x to $u \in \mathcal{M}(p, p')$ passing through $x \in S_{p'} \cap \sigma_p(\partial B^k_{r_p})$, and put $\sharp \mathcal{M}(p, p') := \sum_{x \in S_{p'} \cap \sigma_p(\partial B^k_{r_p})} \epsilon_x$, which is nothing but the intersection number of $S_{p'}$ and $\sigma_p(\partial B^k_{r_p})$.

Similarly, for $p \in Cr_k(f)$ and $\gamma \in Cr^+_{k-1}(f_N)$, we define $\mathcal{M}(p,\gamma)$ to be the set of the unparameterized gradient trajectories $u : \mathbb{R} \to M \setminus N$ which satisfies $du/dt = -X_f \circ u$ with $\lim_{t\to-\infty} u(t) = p$ and $\lim_{t\to\infty} u(t) = (\gamma, 0) \in N \times \{0\} \subset M$; and we define $\sharp\mathcal{M}(p,\gamma)$ to be the intersection number of S_γ and $\sigma_p(\partial B^k_{r_p})$; and moreover, for $\delta \in Cr^-_{k-1}(f_N)$ and $p \in Cr_{k-1}(f)$, we define $\mathcal{M}(p,\gamma)$ to be the set of the unparameterized gradient trajectories $u : \mathbb{R} \to M \setminus N$ which satisfies $du/dt = -X_f \circ u$ with $\lim_{t\to-\infty} u(t) = (\delta, 0) \in N \times \{0\} \subset M$ and $\lim_{t\to\infty} u(t) = p$; and for $\delta \in Cr^-_{k-1}(f_{N_i})$ and $\gamma \in Cr^+_{k-1}(f_{N_j})$ with $c_i > c_j$, we define $\mathcal{M}(\delta,\gamma)$ to be the set of the unparameterized gradient trajectories $u : \mathbb{R} \to M \setminus N$ which satisfies $du/dt = -X_f \circ u$ with $\lim_{t\to-\infty} u(t) = (\delta, 0) \in N_i \times \{0\} \subset M$ and $\lim_{t\to\infty} u(t) = (\gamma, 0) \in N_j \times \{0\} \subset M$; and we define $\sharp\mathcal{M}(\delta,p)$ and $\sharp\mathcal{M}(\delta,\gamma)$ to be the intersection numbers, similarly.

On the other hand, for $\gamma \in Cr^+_k(f_{N_i})$ and $\gamma' \in Cr^+_{k-1}(f_{N_i})$, we define $\mathcal{N}(\gamma,\gamma')$ to be the set of the unparameterized gradient trajectories $u : \mathbb{R} \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$ with $\lim_{t\to-\infty} u(t) = \gamma$ and $\lim_{t\to\infty} u(t) = \gamma'$. We fix a diffeomorphism $\psi_\gamma : B^k_1 \to U_\gamma$ with $\psi_\gamma(0) = \gamma$. Let $0 < r < 1$. For $x \in S_{\gamma'} \cap \psi_\gamma(\partial B^k_r)$, we define $\epsilon_x := 1$ if the orientations of $T_xS_{\gamma'} \oplus T_x\psi_\gamma(\partial B^k_r)$ and T_xM coincide, and $\epsilon_x := -1$ otherwise. Then we assign ϵ_x to $u \in \mathcal{N}(\gamma,\gamma')$ passing through $x \in S_{\gamma'} \cap \psi_\gamma(\partial B^k_r)$, and put $\sharp\mathcal{N}(\gamma,\gamma') := \sum_{x\in S_{\gamma'}\cap\psi_\gamma(\partial B^k_r)} \epsilon_x$, which is nothing but the intersection number of $S_{\gamma'}$ and $\psi_\gamma(\partial B^k_r)$. Similarly, for $\delta \in Cr^-_k(f_{N_i})$ and $\delta' \in Cr^-_{k-1}(f_{N_i})$, we define $\mathcal{N}(\delta,\delta')$ to be the set of the unparameterized gradient trajectories $u : \mathbb{R} \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$ with $\lim_{t\to-\infty} u(t) = \delta$ and $\lim_{t\to\infty} u(t) = \delta'$; and we define $\sharp\mathcal{N}(\delta,\delta')$ to be the intersection number of $S_{\delta'}$ and $\psi_\delta(\partial B^k_r)$.

For $\gamma \in Cr^+_k(f_{N_i})$ and $\delta \in Cr^-_{k-1}(f_{N_i})$, we define $\mathcal{N}(\gamma,\delta)$ to be the set of the unparameterized gradient trajectories $u : \mathbb{R} \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$ with $\lim_{t\to-\infty} u(t) = \gamma$ and $\lim_{t\to\infty} u(t) = \delta$. For $x \in S_\delta \cap \psi_\gamma(\partial B^k_r) \subset N_i$, let $v_{in} \in T_xM$ be an inward-pointing vector, $\{v_1, \ldots, v_{n-k+1}\}$ an oriented basis of T_xS_δ, and $\{v_{n-k+2}, \ldots, v_n\}$ an oriented basins of $T_x\psi_\gamma(\partial B^k_r)$. We define $\epsilon_x := 1$ if the orientations of $\{v_1, \ldots, v_{n-k}, v_{in}, v_{n-k+2}, \ldots, v_n\}$ and T_xM coincide, and $\epsilon_x := -1$ otherwise. Then we assign ϵ_x to $u \in \mathcal{N}(\gamma,\delta)$ passing through $x \in S_\delta \cap \psi_\gamma(\partial B^k_r)$, and put $\sharp\mathcal{N}(\gamma,\delta) := \sum_{x\in S_\delta\cap\psi_\gamma(\partial B^k_r)} \epsilon_x$.

We remark that, for $\delta \in Cr^-_k(f_{N_i})$ and $\gamma \in Cr^+_{k-1}(f_{N_i})$, there is no map $u : \mathbb{R} \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$ with $\lim_{t\to-\infty} u(t) = \delta$ and $\lim_{t\to\infty} u(t) = \gamma$ since $f_{N_i}(\delta) < 0 < f_{N_i}(\gamma)$.

7. Morse homology of manifolds with boundary

Finally we recall our Morse homology of manifolds with boundary, introduced in [1], and re-certify that the Morse homology is isomorphic to the absolute singular homology through connecting homomorphisms.

Let M be an n-dimensional oriented compact manifold with boundary N as before, and g and f a cone end Riemannian metric and a cone end Morse–Smale function on $M \setminus N$, respectively. Moreover, let $M^{-1} = \emptyset \subset M^0 \subset M^1 \subset \cdots \subset M^n = M$ be the handlebody decomposition of M as in Theorem 4.3. Then, owing to the conditions of M^k,

$$H_l(M^k, M^{k-1}; \mathbb{Z}) = \begin{cases} \bigoplus_{p \in Cr_k(f)} \mathbb{Z}[\sigma_p] \oplus \bigoplus_{\gamma \in Cr_k^+(f_N)} \mathbb{Z}[\sigma_\gamma], & l = k, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $[\sigma_p]$ and $[\sigma_\gamma]$ are the relative cycles $\sigma_p : (\overline{B^k_{r_p}}, \partial B^k_{r_p}) \to (M^k, M^{k-1})$ and $\sigma_\gamma : (\overline{B^k_{r_\gamma}} \cup (\partial B^k_{r_\gamma} \times [0, T_\gamma]), \partial B^k_{r_\gamma} \times \{T_\gamma\}) \to (M^k, M^{k-1})$.

We denote by $\delta_k : H_k(M^k, M^{k-1}; \mathbb{Z}) \to H_{k-1}(M^{k-1}, M^{k-2}; \mathbb{Z})$ the connecting homomorphism. The connecting homomorphisms satisfy $\delta_{k-1} \circ \delta_k = 0$, and we obtain a chain complex $(H_*(M^*, M^{*-1}; \mathbb{Z}), \delta_*)$. Because of (4) we can prove that the homology of $(H_*(M^*, M^{*-1}; \mathbb{Z}), \delta_*)$ is isomorphic to the absolute singular homology of M in the same way as CW decompositions.

On the other hand, for a relative cycle $[\sigma : (\Sigma, \partial\Sigma) \to (M^k, M^{k-1})]$, the connecting homomorphism δ_k can be written as

$$\delta_k[\sigma : (\Sigma, \partial\Sigma) \to (M^k, M^{k-1})] = [\sigma : (\partial\Sigma, \emptyset) \to (M^{k-1}, M^{k-2})].$$

Since S_p and the image of σ_p intersect transversely and positively at p, we may think of S_p as the dual base of $[\sigma_p]$, and similarly since S_γ and the image of σ_γ intersect transversely and positively at $\rho_\varepsilon(\gamma)$, we may think of S_γ as the dual base of $[\sigma_\gamma]$. Hence δ_k can be written as

$$\begin{aligned} &\delta_k[\sigma : (\Sigma, \partial\Sigma) \to (M^k, M^{k-1})] \\ &= \sum_{p \in Cr_{k-1}(f)} \sharp(S_p \cap \sigma(\partial\Sigma))[\sigma_p] + \sum_{\gamma \in Cr^+_{k-1}(f_N)} \sharp(S_\gamma \cap \sigma(\partial\Sigma))[\sigma_\gamma], \end{aligned}$$

where $\sharp(S_p \cap \sigma(\partial\Sigma))$ and $\sharp(S_\gamma \cap \sigma(\partial\Sigma))$ are the intersection numbers of S_p and $\sigma(\partial\Sigma)$, and S_γ and $\sigma(\partial\Sigma)$, respectively. This description was essentially given by J. Milnor in [6].

Now we define a free $\mathbb{Z}$ module $CM_k(f)$ by

$$CM_k(f) := \bigoplus_{p \in Cr_k(f)} \mathbb{Z}p \oplus \bigoplus_{\gamma \in Cr_k^+(f_N)} \mathbb{Z}\gamma,$$

which is isomorphic to $H_k(M^k, M^{k-1}; \mathbb{Z})$ by identifying p and γ with the relative cycles $[\sigma_p]$ and $[\sigma_\gamma]$, respectively, and define a linear map $\partial_k : CM_k(f) \to$

$CM_{k-1}(f)$ by, for $p \in Cr_k(f)$,

$$\partial_k p := \sum_{p' \in Cr_{k-1}(f)} \sharp\mathcal{M}(p,p')p' + \sum_{\gamma \in Cr^+_{k-1}(f_N)} \sharp\mathcal{M}(p,\gamma)\gamma,$$

and for $\gamma \in Cr^+_k(f_N)$,

$$\begin{aligned}\partial_k \gamma := & \sum_{p \in Cr_{k-1}(f)} \sum_{\delta \in Cr^-_{k-1}(f_N)} \sharp\mathcal{N}(\gamma,\delta)\sharp\mathcal{M}(\delta,p)p \\ & + \sum_{\gamma' \in Cr^+_{k-1}(f_N)} \sum_{\delta \in Cr^-_{k-1}(f_N)} \sharp\mathcal{N}(\gamma,\delta)\sharp\mathcal{M}(\delta,\gamma')\gamma' \\ & + \sum_{\gamma' \in Cr^+_{k-1}(f_N)} \sharp\mathcal{N}(\gamma,\gamma')\gamma'.\end{aligned}$$

Again our main theorem is that:

Theorem 7.1. *$(CM_*(f), \partial_*)$ is a chain complex, i.e. $\partial_{k-1} \circ \partial_k = 0$, and the homology is isomorphic to the absolute singular homology of M.*

Proof. We already know $CM_k(f) \cong H_k(M^k, M^{k-1}; \mathbb{Z})$, and hence we show that $\partial_k = \delta_k$. For $p \in Cr_k(f)$, by the descriptions of the connecting homomorphisms and the moduli spaces of gradient trajectories,

$$\begin{aligned}&\delta_k[\sigma_p] \\ &= \sum_{p' \in Cr_{k-1}(f)} \sharp(S_{p'} \cap \sigma_p(\partial R^k_{r_p}))[\sigma_{p'}] + \sum_{\gamma \in Cr^+_{k-1}(f_N)} \sharp(S_\gamma \cap \sigma_p(\partial R^k_{r_p}))[\sigma_\gamma] \\ &= \sum_{p' \in Cr_{k-1}(f)} \sharp\mathcal{M}(p,p')[\sigma_{p'}] + \sum_{\gamma \in Cr^+_{k-1}(f_N)} \sharp\mathcal{M}(p,\gamma)[\sigma_\gamma].\end{aligned}$$

Hence $\partial_k p = \delta_k[\sigma_p]$ under the identification of p and γ with $[\sigma_p]$ and $[\sigma_\gamma]$, respectively.

Recall that the relative cycle $\sigma_\gamma : \overline{B^k_{r_\gamma}} \cup (\partial B^k_{r_\gamma} \times [0, T_\gamma]) \to M$ for $\gamma \in Cr^+_k(f_N)$ is given by $\sigma_\gamma(x) := \rho_\varepsilon \circ \psi_\gamma(x)$ if $x \in \overline{B^k_{r_\gamma}}$, and $\sigma_\gamma(x,t) := \varphi_t(\rho_\varepsilon \circ \psi_\gamma(x))$ if $(x,t) \in \partial B^k_{r_\gamma} \times [0, T_\gamma]$. Then, for $\gamma \in Cr^+_k(f_N)$, $\delta_k[\sigma_\gamma]$ can be written as

$$\begin{aligned}\delta_k[\sigma_\gamma] = & \sum_{p \in Cr_{k-1}(f)} \sharp(S_p \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})[\sigma_p] \\ & + \sum_{\gamma' \in Cr^+_{k-1}(f_N)} \sharp(S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})[\sigma_{\gamma'}].\end{aligned}$$

Let $\gamma \in Cr^+_k(f_{N_i})$. For $p \in Cr_{k-1}(f)$, an intersection point $x \in S_p \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$ corresponds to a pair of

- a gradient trajectory $u : (-\infty, r_\gamma] \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$ with $\lim_{t \to -\infty} u(t) = \gamma$, and

- a gradient trajectory $v : [0, \infty) \to M \setminus N$ passing through x which satisfies $dv/dt = -X_f \circ v$ with $v(0) = \rho_\varepsilon(u(r_\gamma))$ and $\lim_{t\to\infty} v(t) = p$.

As $\varepsilon \to 0$, v breaks into two pieces:

- one is a gradient trajectory $u' : [0, \infty) \to N_i$ which satisfies $du'/dt = -X_{f_{N_i}} \circ u'$ with $u'(0) = u(r_\gamma)$ and $\lim_{t\to-\infty} u(t) = \delta \in Cr^-_{k-1}(f_{N_i})$, and
- the other is an unparameterized gradient trajectory $v' : \mathbb{R} \to M \setminus N$ which satisfies $dv'/dt = -X_f \circ v'$ with $\lim_{t\to-\infty} v'(t) = (\delta, 0) \in N_i \times \{0\}$ and $\lim_{t\to\infty} v'(t) = p$.

Note that u and u' give an unparameterized gradient trajectory $u'' : \mathbb{R} \to N_i$ which satisfies $du''/dt = -X_{f_{N_i}} \circ u''$ with $\lim_{t\to-\infty} u''(t) = \gamma$ and $\lim_{t\to\infty} u''(t) = \delta$. Then an intersection point $x \in S_p \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$ gives $(u'', v') \in \mathcal{N}(\gamma, \delta) \times \mathcal{M}(\delta, p)$. Conversely, by the gluing analysis, $(u'', v') \in \mathcal{N}(\gamma, \delta) \times \mathcal{M}(\delta, p)$ gives $x \in S_p \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$. Hence

$$\sharp(S_p \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\}) = \sharp\mathcal{N}(\gamma, \delta)\sharp\mathcal{M}(\delta, p).$$

Let $\gamma \in Cr^+_k(f_{N_i})$. If $\gamma' \in Cr^+_{k-1}(f_{N_j})$ with $c_i > c_j$, an intersection point $x \in S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$ corresponds to a pair of

- a gradient trajectory $u : (-\infty, r_\gamma] \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$ with $\lim_{t\to-\infty} u(t) = \gamma$, and
- a gradient trajectory $v : [0, \infty) \to M \setminus N$ passing through x which satisfies $dv/dt = -X_f \circ v$ with $v(0) = \rho_\varepsilon(u(r_\gamma))$ and $\lim_{t\to\infty} v(t) = (\gamma', 0) \in N_j \times \{0\}$.

As $\varepsilon \to 0$, v breaks into two pieces:

- one is a gradient trajectory $u' : [0, \infty) \to N_i$ which satisfies $du'/dt = -X_{f_{N_i}} \circ u'$ with $u'(0) = u(r_\gamma)$ and $\lim_{t\to\infty} u(t) = \delta \in Cr^-_{k-1}(f_{N_i})$, and
- the other is an unparameterized gradient trajectory $v' : \mathbb{R} \to M \setminus N$ which satisfies $dv'/dt = -X_f \circ v'$ with $\lim_{t\to-\infty} v'(t) = (\delta, 0) \in N_i \times \{0\}$ and $\lim_{t\to\infty} v'(t) = (\gamma', 0) \in N_j \times \{0\}$.

Note that u and u' give an unparameterized gradient trajectory $u'' : \mathbb{R} \to N_i$ which satisfies $du''/dt = -X_{f_{N_i}} \circ u''$ with $\lim_{t\to-\infty} u''(t) = \gamma$ and $\lim_{t\to\infty} u''(t) = \delta$. Then an intersection point $x \in S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$ gives $(u'', v') \in \mathcal{N}(\gamma, \delta) \times \mathcal{M}(\delta, \gamma')$. Conversely, by the gluing analysis, $(u'', v') \in \mathcal{N}(\gamma, \delta) \times \mathcal{M}(\delta, \gamma')$ gives $x \in S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$. Hence

$$\sharp(S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\}) = \sharp\mathcal{N}(\gamma, \delta)\sharp\mathcal{M}(\delta, \gamma').$$

If $\gamma' \in Cr^+_{k-1}(f_{N_i})$, an intersection point $x \in S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$ corresponds to a pair of

- a gradient trajectory $u : (-\infty, r_\gamma] \to N_i$ which satisfies $du/dt = -X_{f_{N_i}} \circ u$ with $\lim_{t\to-\infty} u(t) = \gamma$, and

- a gradient trajectory $v : [0,\infty) \to M \setminus N$ passing through x which satisfies $dv/dt = -X_f \circ v$ with $v(0) = \rho_\varepsilon(u(r_\gamma))$ and $\lim_{t\to\infty} v(t) = (\gamma', 0) \in N_i \times \{0\}$.

This time, as $\varepsilon \to 0$, v converges to a gradient trajectory $v' : [0,\infty) \to N_i$ which satisfies $dv'/dt = -X_{f_{N_i}} \circ v'$ with $v'(0) = u(r_\gamma)$ and $\lim_{t\to\infty} v'(t) = \gamma'$ because, if v converged to a pair of the following two maps:

- a gradient trajectory $u' : [0,\infty) \to N_i$ which satisfies $du'/dt = -X_{f_{N_i}} \circ u'$ with $u'(0) = u(r_\gamma)$ and $\lim_{t\to\infty} u(t) = \delta \in Cr^-_{k-1}(f_{N_i})$, and
- an unparameterized gradient trajectory $v'' : \mathbb{R} \to M \setminus N$ which satisfies $dv''/dt = -X_f \circ v''$ with $\lim_{t\to-\infty} v''(t) = (\delta, 0) \in N_i \times \{0\}$ and $\lim_{t\to\infty} v''(t) = (\gamma', 0) \in N_i \times \{0\}$,

the existence of such v'' contradicts Lemma 2.5. Note that u and v' give an unparameterized gradient trajectory $u'' : \mathbb{R} \to N_i$ which satisfies $du''/dt = -X_{f_{N_i}} \circ u''$ with $\lim_{t\to-\infty} u''(t) = \gamma$ and $\lim_{t\to\infty} u''(t) = \gamma'$. Then an intersection point $x \in S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$ gives $u'' \in \mathcal{N}(\gamma, \gamma')$; and conversely, $u'' \in \mathcal{N}(\gamma, \gamma')$ gives $x \in S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\})$. Hence

$$\sharp(S_{\gamma'} \cap \sigma_\gamma(\partial B^k_{r_\gamma} \times \{T_\gamma\}) = \sharp\mathcal{N}(\gamma, \gamma').$$

Therefore, we obtain

$$\begin{aligned} \delta_k[\sigma_\gamma] &= \sum_{p \in Cr_{k-1}(f)} \sum_{\delta \in Cr^-_{k-1}(f_N)} \sharp\mathcal{N}(\gamma,\delta)\sharp\mathcal{M}(\delta,p)[\sigma_p] \\ &\quad + \sum_{\gamma' \in Cr^+_{k-1}(f_N)} \sum_{\delta \in Cr^-_{k-1}(f_N)} \sharp\mathcal{N}(\gamma,\delta)\sharp\mathcal{M}(\delta,\gamma')[\sigma_{\gamma'}] \\ &\quad + \sum_{\gamma' \in Cr^+_{k-1}(f_N)} \sharp\mathcal{N}(\gamma,\gamma')[\sigma_{\gamma'}], \end{aligned}$$

which implies that $\partial_k\gamma = \delta_k[\sigma_\gamma]$ under the identification of p and γ with $[\sigma_p]$ and $[\sigma_\gamma]$, respectively. □

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