GLUING CONSTRUCTIONS OF PSEUDO-HOLOMORPHIC DISCS AND DESINGULARIZATION

MANABU AKAHO

1. Introduction

Let M be a symplectic manifold with convex boundary N and $L \subset M$ a Lagrangian submanifold with Legendrian boundary $\Lambda \subset N$, and let M' be a symplectic manifold with concave boundary N and $L' \subset M'$ a Lagrangian submanifold with Legendrian boundary $\Lambda \subset N$. Then we construct a symplectic manifold $M\sharp_\rho M' = M \cup ([-\rho, \rho] \times N) \cup M'$ and a Lagrangian submanifold $L\sharp_\rho L' = L \cup ([-\rho, \rho] \times \Lambda) \cup L'$, for some $\rho > 0$. Choose nice almost complex structures on $M \cup [0, \infty) \times N$, $(-\infty, 0] \times N \cup M'$ and $M\sharp_\rho M'$, and we can construct a pseudo-holomorphic disc $w : D^2 = \{z \in \mathbf{C} | |z| \leq 1\} \to M\sharp_\rho M'$ with $w(\partial D^2) \subset L\sharp_\rho L'$ by gluing the following two punctured pseudo-holomorphic discs: one is $u : D^2 \setminus \{1\} \to M \cup [0, \infty) \times N$ such that $u(\partial D^2 \setminus \{1\}) \subset L \cup [0, \infty) \times \Lambda$ and the puncture converges to a Reeb chord in $\{\infty\} \times N$, and the other is $v : D^2 \setminus \{-1\} \to (-\infty, 0] \times N \cup M'$ such that $v(\partial D^2 \setminus \{-1\}) \subset (-\infty, 0] \times \Lambda \cup L'$ and the puncture converges to the Reeb chord in $\{-\infty\} \times N$. Our gluing technique is an improvement on that of Floer [1].

2. Contact and Symplectic Preliminaries

Let N be a smooth manifold of dimension $2n + 1$. We call a 1-form λ on N a *contact form* if $\lambda \wedge (d\lambda)^n$ is a volume form on N. A *contact structure* ξ is the $2n$ dimensional plane field on N defined by $\lambda|_\xi = 0$ and a *Reeb vector field* X_λ is the vector field on N defined by $\lambda(X_\lambda) = 1$ and $d\lambda(X_\lambda, \cdot) = 0$. It is easy to see that $d\lambda|_\xi$ is nondegenerate and there exist complex structures J_ξ on ξ, i.e., $J_\xi \in \mathrm{End}(\xi)$ and $J_\xi^2 = -1$, such that $g_\xi(\cdot, \cdot) = d\lambda(\cdot, J_\xi \cdot)$ is an inner product on ξ.

Consider $\mathbf{R} \times N$ and denote by θ the standard coordinate on the first factor. Then $d(e^\theta \lambda)$ is a symplectic form on $\mathbf{R} \times N$, and we call $(\mathbf{R} \times N, d(e^\theta \lambda))$ the *symplectization* of (N, λ). Let $p_2 : \mathbf{R} \times N \to N$ be the projection $p_2(\theta, x) = x$. We simply denote $p_2^*\lambda$, $p_2^*\xi$, $p_2^* X_\lambda$ and $p_2^* J_\xi$ by λ, ξ, X_λ and J_ξ, respectively. Then we define the almost complex structure J on $\mathbf{R} \times N$ by

- $Jv = J_\xi v$, for $v \in \xi$,
- $J\frac{\partial}{\partial\theta} = X_\lambda$ and $JX_\lambda = -\frac{\partial}{\partial\theta}$.

Let $\Lambda \subset N$ be a submanifold. We call Λ *Legendrian* if $\dim \Lambda = n$ and $\lambda|_{T\Lambda} = 0$. A map $\gamma : [0, T] \to N$ is called a *Reeb chord* if $\dot{\gamma} = X_\lambda$ with $\gamma(0)$ and $\gamma(T) \in \Lambda$, for some $T > 0$.

Let (M, ω) be a noncompact symplectic manifold. Suppose that there exists $K \subset M$ such that $(M \setminus K, \omega)$ is symplectically isomorphic to $((R, \infty) \times N, d(e^\theta \lambda))$, for some $R \in \mathbf{R}$. We call such an end *convex*. We remark that there exist almost

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complex structures $J \in \mathrm{End}(TM)$ such that $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Riemannian metric on M and

- $Jv = J_\xi v$, for $v \in \xi$,
- $J\frac{\partial}{\partial\theta} = X_\lambda$ and $JX_\lambda = -\frac{\partial}{\partial\theta}$

on the convex end.

Let $L \subset M$ be a properly embedded noncompact Lagrangian submanifold whose restriction on the convex end is of the form $(R,\infty) \times \Lambda$.

Similarly, let (M', ω') be a noncompact symplectic manifold. Suppose that there exists $K' \subset M'$ such that $(M' \setminus K', \omega)$ is symplectically isomorphic to $((-\infty, R') \times N, d(e^\theta \lambda))$, for some $R' \in \mathbf{R}$. We call such an end *concave*. We remark that there exist almost complex structures $J' \in \mathrm{End}(TM')$ such that $g_{J'}(\cdot,\cdot) = \omega'(\cdot, J'\cdot)$ is a Riemannian metric on M' and

- $J'v = J_\xi v$, for $v \in \xi$,
- $J'\frac{\partial}{\partial\theta} = X_\lambda$ and $J'X_\lambda = -\frac{\partial}{\partial\theta}$

on the concave end.

Let $L' \subset M'$ be a properly embedded noncompact Lagrangian submanifold whose restriction on the concave end is of the form $(-\infty, R') \times \Lambda$.

We assume that $R = R' = 0$ hereafter. Then, for $\rho > 0$, we define $M\sharp_\rho M'$ by $K \cup ((0,\rho] \times N) \cup ([-\rho, 0) \times N) \cup K'$, i.e., we glue $K \cup ((0,\rho] \times N) \subset M$ and $([-\rho,0) \times N) \cup K' \subset M'$ along the boundaries by the natural identification $\{\rho\} \times N$ with $\{-\rho\} \times N$, and define $L\sharp_\rho L'$ by $(L \cap K) \cup ((0,\rho] \times \Lambda) \cup ([-\rho,0) \times \Lambda) \cup (L' \cap K')$. We often identify $((0,\rho] \times N) \cup ([-\rho,0) \times N) \subset M\sharp_\rho M'$ with $(-\rho,\rho) \times N$.

We remark that we can relax the cylindrical end conditions for L and L' into similar ones of asymptotically conical Lagrangian submanifolds and isolated conical singularities of Lagrangian submanifolds as in [5] and [6]. But we put the conditions for L and L' for simplicity.

3. Smooth Maps

Let g be the Reimannian metric $\lambda \otimes \lambda + g_\xi$ on N. Then $J_\xi T_p\Lambda$ is the orthogonal complement to $T_p\Lambda$ in ξ_p, and $\exp^g \circ (\mathrm{id} \oplus J_\xi)$ gives a diffeomorphism from a neighborhood of the zero section $0 \oplus 0_\Lambda \subset \underline{\mathbf{R}} \oplus T\Lambda$ to a neighborhood of $\Lambda \subset N$, where $\underline{\mathbf{R}}$ is the trivial bundle with fiber $\mathbf{R}$ over Λ. Let g_Λ be a Riemannian metric on Λ. The Levi-Civita connection of g_Λ gives the horizontal lift and induces the Riemannian metric $g_{T\Lambda}$ on the total space of $T\Lambda$ such that 0_Λ is totally geodesic. Hence we get a Riemannian metric g_N on N such that $(\exp^g \circ (\mathrm{id} \oplus J_\xi))^* g_N = dz \otimes dz + g_{T\Lambda}$ on a neighborhood of Λ, where z is the fiber coordinate of $\underline{\mathbf{R}}$, and Λ is totally geodesic.

We define the Riemannian metric g on (M,ω) by $g(\cdot,\cdot) = e^{-\theta\beta} g_J$, where $\beta : M \to [0,1]$ is a smooth cutoff function such that $\beta(x) \equiv 1$, for $x \in (1,\infty) \times N$, and $\beta(x) \equiv 0$, for $x \in K$. Then JT_pL is the orthogonal complement to T_pL in T_pM, and $\exp^g \circ J$ gives a diffeomorphism from a neighborhood of the zero section $0_L \subset TL$ to a neighborhood of $L \subset M$. Let g_L be a Riemannian metric on L such that g_L is of the form $d\theta \otimes d\theta + p_2^* g_\Lambda$ on $(0,\infty) \times \Lambda$. The Levi-Civita connection of g_L gives the horizontal lift and induces the Riemannian metric g_{TL} on the total space of TL such that 0_L is totally geodesic. Hence we get a Riemannian metric g_M on M of the form $d\theta \otimes d\theta + p_2^* g_N$ on $(0,\infty) \times N$, and L is totally geodesic.

Define $\Theta = \{z \in \mathbf{C} | \mathrm{Im} z \geq 0\}$. For a Reeb chord γ, $C_0^\infty(\Theta; \gamma)$ is the set of the smooth maps $\mu : \Theta \to M$ which satisfy the following conditions:

- All derivatives of μ have continuous extensions to Θ.
- $\mu(\partial\Theta) \subset L$.
- For some R_μ, $\mu(z) = (\frac{T}{\pi}\log|z|, \gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))$ when $|z| > R_\mu$.

For $\mu \in C_0^\infty(\Theta;\gamma)$, we define $C_0^\infty(\mu^*TM)$ by the set of the smooth sections $\zeta : \Theta \to \mu^*TM$ which satisfy the following conditions:

- All derivatives of ζ have continuous extensions to Θ.
- $\zeta(\partial\Theta) \subset \mu^*TL$.
- For some R_ζ, $\zeta(z) = 0$ when $|z| > R_\zeta$.

Lemma 3.1. *For $\mu \in C_0^\infty(\Theta;\gamma)$ and $\zeta \in C_0^\infty(\mu^*TM)$, $u = \exp_\mu^{g_M}\zeta$ is also in $C_0^\infty(\Theta;\gamma)$.*

Similarly, we define the Riemannian metric g' on (M',ω') by $g'(\cdot,\cdot) = e^{-\theta\beta'}g_{J'}$, where $\beta' : M' \to [0,1]$ is a smooth cutoff function such that $\beta'(x) \equiv 1$ for $x \in (-\infty,-1)\times N$ and $\beta'(x) \equiv 0$ for $x \in K'$. Then $J'T_pL'$ is the orthogonal complement to T_pL' in T_pM', and $\exp^{g'}\circ J'$ gives a diffeomorphism from a neighborhood of the zero section $0_{L'} \subset TL'$ to a neighborhood of $L' \subset M'$. Let $g_{L'}$ be a Riemannian metric on L' such that $g_{L'}$ is of the form $d\theta\otimes d\theta + p_2^*g_\Lambda$ on $(-\infty,0)\times\Lambda$. The Levi-Civita connection of $g_{L'}$ gives the horizontal lift and induces the Riemannian metric $g_{TL'}$ on the total space of TL' such that $0_{L'}$ is totally geodesic. Hence we get a Riemannian metric $g_{M'}$ on M' of the form $d\theta\otimes d\theta + p_2^*g_N$ on $(-\infty,0)\times N$, and L' is totally geodesic.

Define $\Xi = (\{z \in \mathbf{C} | \mathrm{Im} z \geq 0\} \cup \{\infty\}) \setminus \{0\}$. For a Reeb chord γ, $C_0^\infty(\Xi;\gamma)$ is the set of the smooth maps $\nu : \Xi \to M'$ which satisfy the following conditions:

- All derivatives of ν have continuous extensions to Ξ.
- $\nu(\partial\Xi) \subset L'$.
- For some R_ν, $\nu(z) = (\frac{T}{\pi}\log|z|, \gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))$ when $|z| < R_\nu$.

For $\nu \in C_0^\infty(\Xi;\gamma)$, we define $C_0^\infty(\nu^*TM')$ by the set of the smooth sections $\eta : \Xi \to \nu^*TM'$ which satisfy the following conditions:

- All derivatives of η have continuous extensions to Ξ.
- $\eta(\partial\Xi) \subset \nu^*TL'$.
- For some R_η, $\eta(z) = 0$ when $|z| < R_\eta$.

Lemma 3.2. *For $\nu \in C_0^\infty(\Xi;\gamma)$ and $\eta \in C_0^\infty(\nu^*TM')$, $v = \exp_\nu^{g_{M'}}\eta$ is also in $C_0^\infty(\Xi;\gamma)$.*

Let $g_{M\sharp_\rho M'}$ be the Riemannian metric on $M\sharp_\rho M'$ such that $g_{M\sharp_\rho M'}|_{K\cup(0,\rho]} = g_M$ and $g_{M\sharp_\rho M'}|_{[-\rho,0)\cup K'} = g_{M'}$. We define $e^{-\rho}\Theta = \{e^{-\rho}a | a \in \Theta\}$, $e^\rho\Xi = \{e^\rho b | b \in \Xi\}$ and $\Delta_\rho = (e^{-\rho}\Theta \sqcup e^\rho\Xi)/\sim$, where $z \sim w$ for $z \in e^{-\rho}\Theta$ and $w \in e^\rho\Xi$ if $z = w$. We remark that Δ_ρ is diffeomorphic to the disc D^2. Then $C^\infty(\Delta_\rho)$ is the set of the smooth maps $\upsilon : \Delta_\rho \to M\sharp_\rho M'$ which satisfy the following conditions:

- All derivatives of υ have continuous extensions to Δ_ρ.
- $\upsilon(\partial\Delta_\rho) \subset L\sharp_\rho L'$.

For $\upsilon \in C^\infty(\Delta_\rho)$, we define $C^\infty(\upsilon^*T(M\sharp_\rho M'))$ by the set of the smooth sections $\chi : \Delta_\rho \to \upsilon^*T(M\sharp_\rho M')$ which satisfy the following conditions:

- All derivatives of χ have continuous extensions to Δ_ρ.
- $\chi(\partial\Delta_\rho) \subset \upsilon^*T(L\sharp_\rho L')$.

Lemma 3.3. *For $\upsilon \in C^\infty(\Delta_\rho)$ and $\chi \in C^\infty(\upsilon^* T(M\sharp_\rho M'))$, $v = \exp_\upsilon^{g_{M\sharp_\rho M'}} \chi$ is also in $C^\infty(\Delta_\rho)$.*

4. Banach Manifolds

Let $p > 2$ and $\sigma \in \mathbf{R}$. For $\mu \in C_0^\infty(\Theta;\gamma)$ and $\zeta \in C_0^\infty(\mu^* TM)$, we define

$$\|\zeta\|_{W_\sigma^{1,p}(\Theta)} = \left(\int_\Theta (|\zeta|^p + |\nabla\zeta|^p)\alpha^\sigma(|z|)dxdy \right)^{1/p},$$

where $|\cdot|$ is the norm with respect to g_M, ∇ is the Levi-Civita connection of g_M and $\alpha^\sigma : [0,\infty) \to \mathbf{R}_{>0}$ is a weight function such that $\alpha^\sigma(r) = r^{-2+\sigma}$, for $r \geq 1$. We remark that, through $(s,t) = (\log|z|, \frac{1}{i}\log\frac{z}{|z|})$,

$$\int_{\Theta\cap\{|z|>1\}} (|\zeta|^p + |\nabla\zeta|^p)\alpha^\sigma(|z|)dxdy = \int_{(0,\infty)\times[0,\pi]} (|\zeta|^p + |\nabla\zeta|^p)e^{\sigma s}dsdt.$$

Let $W_\sigma^{1,p}(\mu^* TM)$ be the completion of $C_0^\infty(\mu^* TM)$ by $\|\cdot\|_{W_\sigma^{1,p}(\Theta)}$ and define

$$W_\sigma^{1,p}(\Theta;\gamma) = \left\{ \exp_\mu^{g_M} \zeta \;\middle|\; \mu \in C_0^\infty(\Theta;\gamma), \zeta \in W_\sigma^{1,p}(\mu^* TM) \right\}.$$

From the Sobolev embedding theorem, $u \in W_\sigma^{1,p}(\Theta;\gamma)$ satisfies

- $u : \Theta \to M$ is continuous,
- $u(\partial\Theta) \subset L$,
- u asymptotically approaches $(\frac{T}{\pi}\log|z|, \gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))$ at $z = \infty$.

For $u = \exp_\mu^{g_M} \zeta \in W_\sigma^{1,p}(\Theta;\gamma)$, we define

$$T_u W_\sigma^{1,p}(\Theta;\gamma) = W_\sigma^{1,p}(\mu^* TM).$$

Lemma 4.1. *$W_\sigma^{1,p}(\Theta;\gamma)$ is a Banach manifold whose tangent space at u is $T_u W_\sigma^{1,p}(\Theta;\gamma)$.*

For $\mu \in C_0^\infty(\Theta;\gamma)$, we denote by $L_\sigma^p(\wedge^{0,1}\Theta \otimes \mu^* TM)$ the set of the measurable sections of $\wedge^{0,1}\Theta \otimes \mu^* TM$ for which the norm

$$\|\zeta\|_{L_\sigma^p(\Theta)} = \left(\int_\Theta |\zeta|^p \alpha^\sigma(|z|)dxdy \right)^{1/p}$$

is finite. Moreover, for $u = \exp_\mu^{g_M} \zeta \in W_\sigma^{1,p}(\Theta;\gamma)$, we define

$$L_\sigma^p(\Theta;\gamma)_u = L_\sigma^p(\wedge^{0,1}\Theta \otimes \mu^* TM)$$

and

$$L_\sigma^p(\Theta;\gamma) = \bigcup_{u \in W_\sigma^{1,p}(\Theta;\gamma)} L_\sigma^p(\Theta;\gamma)_u.$$

Lemma 4.2. *$L_\sigma^p(\Theta;\gamma)$ is a Banach space bundle whose fiber over u is $L_\sigma^p(\Theta;\gamma)_u$.*

For $\nu \in C_0^\infty(\Xi;\gamma)$ and $\eta \in C_0^\infty(\nu^* TM')$, we define

$$\|\eta\|_{W_\sigma^{1,p}(\Xi)} = \left(\int_\Xi (|\eta|^p + |\nabla\eta|^p)\frac{\alpha^\sigma(|z|^{-1})}{|z|^4}dxdy \right)^{1/p},$$

where $|\cdot|$ is the norm with respect to $g_{M'}$ and ∇ is the Levi-Civita connection of $g_{M'}$. We remark that, through $(s,t) = (\log|z|, \frac{1}{i}\log\frac{z}{|z|})$,

$$\int_{\Xi\cap\{|z|<1\}} (|\eta|^p + |\nabla\eta|^p)\frac{\alpha^\sigma(|z|^{-1})}{|z|^4}dxdy = \int_{(-\infty,0)\times[0,\pi]} (|\eta|^p + |\nabla\eta|^p)e^{-\sigma s}dsdt.$$

Let $W^{1,p}_\sigma(\nu^*TM')$ be the completion of $C^\infty_0(\nu^*TM')$ by $\|\cdot\|_{W^{1,p}_\sigma(\Xi)}$ and define

$$W^{1,p}_\sigma(\Xi;\gamma)=\left\{\exp^{g_{M'}}_\nu \eta \;\middle|\; \nu\in C^\infty_0(\Xi;\gamma),\eta\in W^{1,p}_\sigma(\nu^*TM')\right\}.$$

From the Sobolev embedding theorem, $v\in W^{1,p}_\sigma(\Xi;\gamma)$ satisfies

- $v:\Xi\to M'$ is continuous,
- $v(\partial\Xi)\subset L'$,
- v asymptotically approaches $(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))$ near $z=0$.

For $v=\exp^{g_{M'}}_\nu\eta\in W^{1,p}_\sigma(\Xi;\gamma)$, we define

$$T_vW^{1,p}_\sigma(\Xi;\gamma)=W^{1,p}_\sigma(\nu^*TM').$$

Lemma 4.3. *$W^{1,p}_\sigma(\Xi;\gamma)$ is a Banach manifold whose tangent space at v is $T_vW^{1,p}_\sigma(\Xi;\gamma)$.*

For $\nu\in C^\infty_0(\Xi;\gamma)$, we denote by $L^p_\sigma(\wedge^{0,1}\Xi\otimes\nu^*TM')$ the set of the measurable sections of $\wedge^{0,1}\Xi\otimes\nu^*TM'$ for which the norm

$$\|\eta\|_{L^p_\sigma(\Xi)}=\left(\int_\Xi|\eta|^p\frac{\alpha^\sigma(|z|^{-1})}{|z|^4}dxdy\right)^{1/p}$$

is finite. Moreover, for $v=\exp^{g_{M'}}_\nu\eta\in W^{1,p}_\sigma(\Xi;\gamma)$, we define

$$L^p_\sigma(\Xi;\gamma)_v=L^p_\sigma(\wedge^{0,1}\Xi\otimes\nu^*TM')$$

and

$$L^p_\sigma(\Xi;\gamma)=\bigcup_{v\in W^{1,p}_\sigma(\Xi;\gamma)}L^p_\sigma(\Xi;\gamma)_v.$$

Lemma 4.4. *$L^p_\sigma(\Xi;\gamma)$ is a Banach space bundle whose fiber over v is $L^p_\sigma(\Xi;\gamma)_v$.*

For $v\in C^\infty(\Delta_\rho)$ and $\chi\in C^\infty(v^*T(M\sharp_\rho M'))$, we define

$$\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)}=\left(\int_{\Delta_\rho}(|\chi|^p+|\nabla\chi|^p)\beta^\sigma_\rho(|z|)dxdy\right)^{1/p},$$

where $|\cdot|$ is the norm with respect to $g_{M\sharp_\rho M'}$, ∇ is the Levi-Civita connection of $g_{M\sharp_\rho M'}$ and $\beta^\sigma_\rho:[0,\infty]\to\mathbf{R}_{>0}$ is the weight function defined by

$$\beta^\sigma_\rho(|z|)=\begin{cases}\alpha^\sigma(e^\rho|z|)e^{2\rho}, & \text{for } |z|\le 1,\\ \frac{\alpha^\sigma(|e^{-\rho}z|^{-1})}{|e^{-\rho}z|^4}e^{-2\rho}, & \text{for } |z|>1.\end{cases}$$

We remark that, through $(s,t)=(\log|z|,\frac{1}{i}\log\frac{z}{|z|})$,

$$\int_{\Delta_\rho\cap\{e^{-\rho}<|z|\le1\}}(|\chi|^p+|\nabla\chi|^p)\beta^\sigma_\rho(|z|)dxdy=\int_{(-\rho,0]\times[0,\pi]}(|\chi|^p+|\nabla\chi|^p)e^{\sigma(s+\rho)}dsdt$$

and

$$\int_{\Delta_\rho\cap\{1<|z|<e^\rho\}}(|\chi|^p+|\nabla\chi|^p)\beta^\sigma_\rho(|z|)dxdy=\int_{(0,\rho)\times[0,\pi]}(|\chi|^p+|\nabla\chi|^p)e^{-\sigma(s-\rho)}dsdt.$$

Let $W^{1,p}_\sigma(v^*T(M\sharp_\rho M'))$ be the completion of $C^\infty(v^*T(M\sharp_\rho M'))$ by $\|\cdot\|_{W^{1,p}_\sigma(\Delta_\rho)}$ and define

$$W^{1,p}_\sigma(\Delta_\rho)=\left\{\exp^{g_{M\sharp_\rho M'}}_v\chi \;\middle|\; v\in C^\infty(\Delta_\rho),\chi\in W^{1,p}_\sigma(v^*T(M\sharp_\rho M'))\right\}.$$

From the Sobolev embedding theorem, $w \in W^{1,p}_\sigma(\Delta_\rho)$ is continuous with $w(\partial\Delta_\rho) \subset L\sharp_\rho L'$. For $w = \exp_\upsilon^{g_{M\sharp_\rho M'}} \chi \in W^{1,p}_\sigma(\Delta_\rho)$, we define

$$T_w W^{1,p}_\sigma(\Delta_\rho) = W^{1,p}_\sigma(\upsilon^* T(M\sharp_\rho M')).$$

Lemma 4.5. *$W^{1,p}_\sigma(\Delta_\rho)$ is a Banach manifold whose tangent space at w is $T_w W^{1,p}_\sigma(\Delta_\rho)$.*

For $\upsilon \in C^\infty(\Delta_\rho)$, we denote by $L^p_\sigma(\wedge^{0,1}\Delta_\rho \otimes \upsilon^* T(M\sharp_\rho M'))$ the set of the measurable sections of $\wedge^{0,1}\Delta_\rho \otimes \upsilon^* T(M\sharp_\rho M')$ for which the norm

$$\|\chi\|_{L^p_\sigma(\Delta_\rho)} = \left(\int_{\Delta_\rho} |\chi|^p \beta^\sigma_\rho(|z|) dxdy \right)^{1/p}$$

is finite. Moreover, for $w = \exp_\upsilon^{g_{M\sharp_\rho M'}} \chi \in W^{1,p}_\sigma(\Delta_\rho)$, we define

$$L^p_\sigma(\Delta_\rho)_w = L^p_\sigma(\wedge^{0,1}\Delta_\rho \otimes \upsilon^* T(M\sharp_\rho M'))$$

and

$$L^p_\sigma(\Delta_\rho) = \bigcup_{w \in W^{1,p}_\sigma(\Delta_\rho)} L^p_\sigma(\Delta_\rho)_w.$$

Lemma 4.6. *$L^p_\sigma(\Delta_\rho)$ is a Banach space bundle whose fiber over w is $L^p_\sigma(\Delta_\rho)_w$.*

5. Pseudo-holomorphic Discs

For $u \in W^{1,p}_\sigma(\Theta;\gamma)$, we define the *Cauchy-Riemann operator* by

$$\overline{\partial}_J(u) = \frac{1}{2}\left(du + J(u) \circ du \circ j\right) \in L^p_\sigma(\Theta;\gamma)_u,$$

where j is the standard complex structure on Θ. We may think of $\overline{\partial}_J$ as a section of $L^p_\sigma(\Theta;\gamma)$ [7]. Given $\zeta \in T_u W^{1,p}_\sigma(\Theta;\gamma)$, let $\Phi_u(\zeta) : u^*TM \to (\exp_u^{g_M} \zeta)^* TM$ denote the bundle isomorphism given by parallel transport along the geodesic $l(t) = \exp_u^{g_M} t\zeta$. Then we define the map $\mathcal{F}_u : T_u W^{1,p}_\sigma(\Theta;\gamma) \to L^p_\sigma(\Theta;\gamma)_u$ by

$$\mathcal{F}_u(\zeta) = \Phi_u(\zeta)^{-1} \overline{\partial}_J(\exp_u^{g_M} \zeta).$$

We denote by D_u the linearized operator $d\mathcal{F}_u(0) : T_u W^{1,p}_\sigma(\Theta;\gamma) \to L^p_\sigma(\Theta;\gamma)_u$. Then

$$D_u\zeta = \frac{1}{2}(\nabla\zeta + J(u) \circ \nabla\zeta \circ j) - \frac{1}{2} J(u)(\nabla_\zeta J)(u)\partial_J(u),$$

where ∇ is the Levi-Civita connection of g_M and $\partial_J(u) = \frac{1}{2}(du - J(u) \circ du \circ j)$. For some $\sigma > 0$, D_u is Fredholm. We sometimes think of D_u on $\Theta \cap \{|z| > 1\}$ as a differential operator on $\{(s,t) \in (0,\infty) \times [0,\pi]\}$ through $(s,t) = (\log|z|, \frac{1}{i}\log\frac{z}{|z|})$.

We call γ *standard* if there exist a tubular neighborhood U of $\gamma([0,T])$ and an immersion $\phi : \{(x_1, y_1, \ldots, x_n, y_n, z) | \sum_{i=1}^n (x_i^2 + y_i^2) < \epsilon, 0 \le z \le T\} \to U$, for some $\epsilon > 0$, such that

- $\phi(\{0\} \times [0,T]) = \gamma([0,T])$ and $\phi^*\lambda = dz + \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$,
- $\phi^{-1}(\Lambda) \cap B = L_0 \cap B$, where $B = \{(x_1, y_1, \ldots, x_n, y_n, 0) | \sum_{i=1}^n (x_i^2 + y_i^2) < \epsilon\}$ and L_0 is a Lagrangian linear subspace in $\{(x_1, y_1, \ldots, x_n, y_n)\}$,
- $\phi^{-1}(\Lambda) \cap B' = L_T \cap B'$, where $B' = \{(x_1, y_1, \ldots, x_n, y_n, T) | \sum_{i=1}^n (x_i^2 + y_i^2) < \epsilon\}$ and L_T is a Lagrangian linear subspace in $\{(x_1, y_1, \ldots, x_n, y_n)\}$.

Then we may choose g_N and J_ξ so that $\nabla_{\dot\gamma} = \frac{\partial}{\partial z}$ and $\gamma^*\nabla J_\xi = 0$. Let $\varphi_t : N \to N$ be the solution of $\frac{d}{dt}\varphi_t = X_\lambda \circ \varphi_t$ and $\varphi_0 = \mathrm{id}$. Write $\overline{\gamma}(t) = \gamma(Tt/\pi)$. We consider the pull-back bundle $\overline{\gamma}^*\xi$ over $[0,\pi]$. Take $\{e_1, e_2, \ldots, e_n\} \subset \xi_{\overline{\gamma}(0)}$ so that $\{e_1, J_\xi e_1, \ldots, e_n, J_\xi e_n\}$ is a basis of $\xi_{\overline{\gamma}(0)}$. Put $e_i(t) = d\varphi_{Tt/\pi} e_i \in \xi_{\overline{\gamma}(t)}$, and then $\nabla_{\dot{\overline{\gamma}}} e_i(t) = 0$ and $\nabla_{\dot{\overline{\gamma}}} J_\xi e_i(t) = 0$. So $\overline{\gamma}^* J_\xi \circ \overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}}$ is represented as $J_0\frac{\partial}{\partial t}$, where J_0 is the standard complex structure on $\mathbf{R}^{2n}$. Since D_u is of the form $\frac{1}{2}(\nabla_{\frac{\partial}{\partial s}} + J(u(s,t))\nabla_{\frac{\partial}{\partial t}}) - \frac{1}{2}J(u)(\nabla J)(u)\partial_J(u)$ on $(0,\infty)\times[0,\pi]$, it asymptotically approaches the differential operator

$$\frac{1}{2}(\frac{\partial}{\partial s} + \overline{\gamma}^* J \circ \overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})$$

as $s \to \infty$.

We call γ *nondegenerate* if $d\varphi_T T_{\gamma(0)}\Lambda$ and $T_{\gamma(T)}\Lambda$ transversally intersect in $\xi_{\gamma(T)}$. Then, if $\overline{\gamma}^* J \circ \overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}}\zeta(t) = 0$ with $\zeta(0) \in \mathbf{R}\frac{\partial}{\partial\theta} \oplus T_{\gamma(0)}\Lambda$ and $\zeta(\pi) \in \mathbf{R}\frac{\partial}{\partial\theta} \oplus T_{\gamma(T)}\Lambda$, we have $\zeta(t) = c\frac{\partial}{\partial\theta}$, for $c \in \mathbf{R}$.

We define $\mathcal{F}_v : T_v W^{1,p}_\sigma(\Xi;\gamma) \to L^p_\sigma(\Xi;\gamma)_v$ and $D_v = d\mathcal{F}_v(0) : T_v W^{1,p}_\sigma(\Xi;\gamma) \to L^p_\sigma(\Xi;\gamma)_v$, for $v \in W^{1,p}_\sigma(\Xi;\gamma)$, and $\mathcal{F}_w : T_w W^{1,p}_\sigma(\Delta_\rho) \to L^p_\sigma(\Delta_\rho)_w$ and $D_w = d\mathcal{F}_w(0) : T_w W^{1,p}_\sigma(\Delta_\rho) \to L^p_\sigma(\Delta_\rho)_w$, for $w \in W^{1,p}_\sigma(\Delta_\rho)$, similarly.

Lemma 5.1. *For $w \in W^{1,p}_\sigma(\Delta_\rho)$, we write $\mathcal{F}_w(\chi) = \mathcal{F}_w(0) + D_w\chi + N_w(\chi)$. Then there exists some constant C depending only on $\|\nabla w\|_{L^p(\Delta_\rho)}$ such that*

$$\|N_w(\chi) - N_w(\chi')\|_{L^p_\sigma(\Delta_\rho)} \le C(\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} + \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)})\|\chi - \chi'\|_{W^{1,p}_\sigma(\Delta_\rho)},$$

for $\chi, \chi' \in T_w W^{1,p}_\sigma(\Delta_\rho)$ with $\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)}, \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-1}$.

Proof. It is done by the Taylor expansion of $\mathcal{F}_w$.

$$\begin{aligned} N_w(\chi) - N_w(\chi') &= \int_0^1 (1-t)\{d^2\mathcal{F}_w(t\chi)(\chi,\chi) - d^2\mathcal{F}_w(t\chi')(\chi',\chi')\}dt \\ &= \int_0^1 (1-t)\{d^2\mathcal{F}_w(t\chi)(\chi,\chi-\chi') + d^2\mathcal{F}_w(t\chi)(\chi,\chi') - \\ &\quad d^2\mathcal{F}_w(t\chi')(\chi,\chi') + d^2\mathcal{F}_w(t\chi')(\chi-\chi',\chi')\}dt \end{aligned}$$

and

$$d^2\mathcal{F}_w(t\chi)(\chi,\chi') - d^2\mathcal{F}_w(t\chi')(\chi,\chi') = \int_0^1 d^3\mathcal{F}_w((1-s)t\chi + st\chi')(t\chi - t\chi', \chi, \chi')ds.$$

Then we can conclude

$$\begin{aligned} &\|N_w(\chi) - N_w(\chi')\|_{L^p_\sigma(\Delta_\rho)} \\ \le\ & C(\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} + \|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)}\|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)} + \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)})\|\chi - \chi'\|_{W^{1,p}_\sigma(\Delta_\rho)}, \end{aligned}$$

where C is some constant depending only on $\|\nabla w\|_{L^p(\Delta_\rho)}$. Take some large C if necessary, and we obtain the inequality as in the lemma. □

We call $u \in W^{1,p}_\sigma(\Theta;\gamma)$ a *punctured pseudo-holomorphic disc* if $\overline{\partial}_J(u) = 0$. Similarly, we define a punctured pseudo-holomorphic disc, for $v \in W^{1,p}_\sigma(\Xi;\gamma)$. If $w \in W^{1,p}_\sigma(\Delta_\rho)$ satisfies $\overline{\partial}_J(w) = 0$, we call w a *pseudo-holomorphic disc.*

6. Gluing Analysis

For simplicity, we assume that, for $u \in W^{1,p}_\sigma(\Theta;\gamma)$, there exists

$$\overline{u} \in W^{1,p}_\sigma((\frac{T}{\pi}\log|z|, \gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))^*T((0,\infty)\times N))$$

such that $u = \exp^{g_M}_{(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\overline{u}$ on $\{z \in \Theta | \log|z| > 0\}$, and, for $v \in W^{1,p}_\sigma(\Xi;\gamma)$, we assume that there exists

$$\overline{v} \in W^{1,p}_\sigma((\frac{T}{\pi}\log|z|, \gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))^*T((-\infty,0)\times N))$$

such that $v = \exp^{g_M}_{(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\overline{v}$ on $\{z \in \Xi | \log|z| < 0\}$. Then we define $u\sharp_\rho v \in W^{1,p}_\sigma(\Delta_\rho)$ by

$$u\sharp_\rho v = \begin{cases} u(e^\rho z), & \text{for } |z| \le e^{-1}, \\ \exp^{g_M}_{(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\beta_u(\log|z|)\overline{u}(e^\rho z), & \text{for } e^{-1} < |z| \le 1, \\ \exp^{g_{M'}}_{(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\beta_v(\log|z|)\overline{v}(e^{-\rho} z), & \text{for } 1 < |z| \le e, \\ v(e^{-\rho}z), & \text{for } |z| > e, \end{cases}$$

where β_u and β_v are smooth cutoff functions such that

$$\beta_u(s) = \begin{cases} 1, & \text{for } s \le -1, \\ 0, & \text{for } s \ge 0, \end{cases} \quad \text{and} \quad \beta_v(s) = \begin{cases} 0, & \text{for } s \le 0, \\ 1, & \text{for } s \ge 1. \end{cases}$$

For $\zeta \in T_u W^{1,p}_\sigma(\Theta;\gamma)$ and $\eta \in T_v W^{1,p}_\sigma(\Xi;\gamma)$, we similarly define $\zeta\sharp_\rho\eta \in T_{u\sharp_\rho v}W^{1,p}_\sigma(\Delta_\rho)$ by

$$\zeta\sharp_\rho\eta = \begin{cases} \zeta(e^\rho z), & \text{for } |z| \le e^{-2}, \\ \beta_u(\log|z|+1)\zeta(e^\rho z), & \text{for } e^{-2} < |z| \le 1, \\ \beta_v(\log|z|-1)\eta(e^{-\rho} z), & \text{for } 1 < |z| \le e^2, \\ \eta(e^{-\rho}z), & \text{for } |z| > e^2. \end{cases}$$

Lemma 6.1. *Let u and v be punctured pseudo-holomorphic discs. For any $\varepsilon > 0$, there exists some constant ρ_0 depending only on ε, u and v such that*

$$\|\overline{\partial}_J(u\sharp_\rho v)\|_{L^p_\sigma(\Delta_\rho)} < \varepsilon,$$

for $\rho > \rho_0$.

Proof. By the definition of $u\sharp_\rho v$, we obtain

$$\begin{aligned} & \|\overline{\partial}_J(u\sharp_\rho v)\|_{L^p_\sigma(\Delta_\rho)} \\ \le\ & \|\overline{\partial}_J(\exp^{g_M}_{(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\beta_u(\log|z|)\overline{u}(e^\rho z)\|_{L^p_\sigma(\Delta_\rho\cap\{e^{-1}<|z|<1\})} \\ & +\|\overline{\partial}_J(\exp^{g_{M'}}_{(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\beta_v(\log|z|)\overline{v}(e^{-\rho}z)\|_{L^p_\sigma(\Delta_\rho\cap\{1<|z|<e\})} \\ \le\ & C(\|\overline{u}\|_{W^{1,p}_\sigma(\Theta\cap\{e^{\rho-1}<|z|<e^\rho\})} + \|\overline{v}\|_{W^{1,p}_\sigma(\Xi\cap\{e^{-\rho}<|z|<e^{-\rho+1}\})}), \end{aligned}$$

where C is some constant depending only on u and v. Hence we obtain ρ_0 as in the lemma. □

Define $\mathrm{sgn} : \mathbf{R} \to \{-1, 0, 1\}$ by

$$\mathrm{sgn}(s) = \begin{cases} -1, & \text{for } s < 0, \\ 0, & \text{for } s = 0, \\ 1, & \text{for } s > 0. \end{cases}$$

Let $\Lambda_0 \subset \mathbf{R}^{2n}$ be the linear subspace corresponding to $T_{\overline{\gamma}(0)}\Lambda \subset \xi_{\overline{\gamma}(0)}$ through the basis $\{e_1(0), J_\xi e_1(0), \dots, e_n(0), J_\xi e_n(0)\}$ and $\Lambda_\pi \subset \mathbf{R}^{2n}$ the linear subspace corresponding to $T_{\overline{\gamma}(\pi)}\Lambda \subset \xi_{\overline{\gamma}(\pi)}$ through the basis $\{e_1(\pi), J_\xi e_1(\pi), \dots, e_n(\pi), J_\xi e_n(\pi)\}$. We remark that Λ_0 and Λ_π intersect transversely in $\mathbf{R}^{2n}$ since γ is nondegenarate. Moreover, we define

$$W^{1,p}(\mathbf{R}\times[0,\pi], \mathbf{R}^{2n}, \Lambda_0, \Lambda_\pi) = \{\chi \in W^{1,p}(\mathbf{R}\times[0,\pi], \mathbf{R}^{2n}) | \chi(0) \in \Lambda_0 \text{ and } \chi(\pi) \in \Lambda_\pi\}$$

and

$$W^{1,p}([0,\pi], \mathbf{R}^{2n}, \Lambda_0, \Lambda_\pi) = \{\chi \in W^{1,p}([0,\pi], \mathbf{R}^{2n}) | \chi(0) \in \Lambda_0 \text{ and } \chi(\pi) \in \Lambda_\pi\}.$$

Lemma 6.2. *If $\sigma > 0$ is small enough, the operator $\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + sgn(s)\frac{\sigma}{p} : W^{1,p}(\mathbf{R}\times [0,\pi], \mathbf{R}^{2n}, \Lambda_0, \Lambda_\pi) \to L^p(\mathbf{R} \times [0,\pi], \mathbf{R}^{2n})$ is bijective, for $1 < p < \infty$.*

Proof. This lemma is a modification of Lemma 2.4 in [8]. We shall give the proof for $p = 2$. The operator $B = J_0 \frac{\partial}{\partial t} + \mathrm{sgn}(s)\frac{\sigma}{p} : W^{1,2}([0,\pi], \mathbf{R}^{2n}, \Lambda_0, \Lambda_\pi) \to L^2([0,\pi], \mathbf{R}^{2n})$ is a self-adjoint operator on the Hilbert space $L^2([0,\pi], \mathbf{R}^{2n})$ with domain $W^{1,2}([0,\pi], \mathbf{R}^{2n}, \Lambda_0, \Lambda_\pi)$. Since Λ_0 and Λ_π intersect transversely, if $\sigma > 0$ is small enough, then 0 is not an eigenvalue of B. Hence there is a splitting $L^2([0,\pi], \mathbf{R}^{2n}) = E^+ \oplus E^-$ into the positive and negative eigenspaces of B. Denote by $P^\pm : L^2([0,\pi], \mathbf{R}^{2n}) \to E^\pm$ the orthogonal projections. Define

$$K(s) = \begin{cases} e^{-Bs}P^+, & \text{for } s > 0, \\ -e^{-Bs}P^-, & \text{for } s \le 0, \end{cases}$$

and $Q : L^2(\mathbf{R} \times [0,\pi], \mathbf{R}^{2n}) \to W^{1,2}(\mathbf{R} \times [0,\pi], \mathbf{R}^{2n})$ by

$$Q\chi(s,t) = \int_{-\infty}^{\infty} K(s-\tau)\chi(\tau,t)d\tau,$$

and Q is the inverse of $\frac{\partial}{\partial s} + B$. In fact

$$Q\chi(s,t) = \int_{-\infty}^{s} e^{-B(s-\tau)}P^+\chi(\tau,t)d\tau - \int_{s}^{\infty} e^{-B(s-\tau)}P^-\chi(\tau,t)d\tau,$$

and we can check $\frac{\partial}{\partial s}Q\chi + BQ\chi = \chi$ and $Q\frac{\partial}{\partial s}\chi + QB\chi = \chi$ directly. The proof for $p > 2$ is the same as the one of Lemma 2.4 in [8]. □

For $\chi \in T_{\overline{\gamma}(0)}(\mathbf{R} \times N)$, we denote by χ^1 the $\mathbf{R}\frac{\partial}{\partial \theta} \oplus \mathbf{R}X_\lambda$ component of χ and by χ^2 the $\xi_{\overline{\gamma}(0)}$ component of χ.

Proposition 6.3. *Let u and v be punctured pseudo-holomorphic discs and $\{(\rho_i, \chi_i)\}_{i=1}^\infty$ a sequence of pairs $\rho_i \in \mathbf{R}$ and $\chi_i \in T_{u\sharp_{\rho_i}v}W^{1,p}_\sigma(\Delta_{\rho_i})$. Suppose that $\rho_i \to \infty$ and that $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = 1$, $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$ and $\chi^1_i(1) = 0$. Then there exists a subsequence $\{(\rho_{i_l}, \chi_{i_l})\}_{l=1}^\infty$ such that*

$$\|\chi_{i_l}\|_{W^{1,p}_\sigma(\Delta_{\rho_{i_l}} \cap \{e^{-3}<|z|<e^3\})} \to 0.$$

Proof. Fix $N > 1$. We may assume that $u\sharp_{\rho_i}v(\Delta_{\rho_i} \cap \{e^{-N} < |z| < e^N\})$ is contained in a tubular neighborhood of $(-N,N) \times \gamma([0,T])$ in $M\sharp_\rho M'$. For $\chi_i : \Delta_{\rho_i} \cap \{e^{-N} < |z| < e^N\} \to (u\sharp_{\rho_i}v)^*T(M\sharp_\rho M')$, we define $\overline{\chi}_i : \Delta_{\rho_i} \cap \{e^{-N} < |z| < e^N\} \to (\frac{T}{\pi}\log|z|, \gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))^*T(M\sharp_\rho M')$ by

$$D\exp^{g_{M\sharp_{\rho_i}M'}}_{(\frac{T}{\pi}\log|z|, \gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\overline{\chi_i} = \chi_i,$$

and we similarly define $\overline{D}_{u\sharp_{\rho_i}v}$ by

$$D\exp^{g_{M\sharp_{\rho_i}M'}}_{(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))}\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi_i}=D_{u\sharp_{\rho_i}v}\chi_i.$$

We remark that $\overline{D}_{u\sharp_{\rho_i}v}\to\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})$ on $\{e^{-N}<|z|<e^N\}$ in the C^0 topology, i.e., if $\overline{D}_{u\sharp_{\rho_i}v}=\frac{1}{2}(a_i\frac{\partial}{\partial s}+b_i\frac{\partial}{\partial t}+c_i)$, then $a_i\to 1$, $b_i\to\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\oplus J_0$ and $c_i\to 0$ in the C^0 topology. Think of $\overline{\chi}_i|_{\Delta_{\rho_i}\cap\{e^{-N}<|z|<e^N\}}$ as a section $\overline{\chi}_i:[-N,N]\times[0,\pi]\to(\frac{T}{\pi}\log|z|,\gamma(\frac{T}{\pi i}\log\frac{z}{|z|}))^*T(M\sharp_\rho M')$ through $(s,t)=(\log|z|,\frac{1}{i}\log\frac{z}{|z|})$. From $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})}=1$, there exists C such that $\|e^{\sigma/p}_{\rho_i}\overline{\chi}_i\|_{W^{1,p}([-N,N]\times[0,\pi])}<C$, where $e^{\sigma/p}_{\rho_i}:\mathbf{R}\to\mathbf{R}_{>0}$ is the function defined by

$$e^{\sigma/p}_{\rho_i}(s)=\begin{cases}e^{\sigma(s+\rho_i)/p}, & \text{for } s\le 0,\\ e^{-\sigma(s-\rho_i)/p}, & \text{for } s>0.\end{cases}$$

Then, by the Rellich's theorem, there exists $\overline{\chi}_N\in L^p([-N,N]\times[0,\pi])$ and a subsequence $\{(\rho_{i_l},\overline{\chi}_{i_l})\}_{l=1}^\infty$ such that $\|\overline{\chi}_N-e^{\sigma/p}_{\rho_{i_l}}\overline{\chi}_{i_l}\|_{L^p([-N,N]\times[0,\pi])}\to 0$. We omit to mention subsequences hereafter. By the Gärding inequality, we have

$$\begin{aligned}&\|e^{\sigma/p}_{\rho_i}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{\chi}_j\|_{W^{1,p}([-N+1,N-1]\times[0,\pi])}\\ \le\;& C(\|\overline{D}_{u\sharp_{\rho_i}v}(e^{\sigma/p}_{\rho_i}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{\chi}_j)\|_{L^p([-N,N]\times[0,\pi])}+\|e^{\sigma/p}_{\rho_i}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{\chi}_j\|_{L^p([-N,N]\times[0,\pi])})\\ \le\;& C(\|e^{\sigma/p}_{\rho_i}\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_j\|_{L^p([-N,N]\times[0,\pi])}+\|e^{\sigma/p}_{\rho_i}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{\chi}_j\|_{L^p([-N,N]\times[0,\pi])}),\end{aligned}$$

where C is a constant depending only on u and v. We already know $\|e^{\sigma/p}_{\rho_i}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{\chi}_j\|_{L^p([-N,N]\times[0,\pi])}\to 0$. And moreover,

$$\begin{aligned}&\|e^{\sigma/p}_{\rho_i}\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_j\|_{L^p([-N,N]\times[0,\pi])}\\ \le\;&\|\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_i\|_{L^p_\sigma(\Delta_{\rho_i}\cap\{e^{-N}<|z|<e^N\})}+\|\overline{D}_{u\sharp_{\rho_j}v}\overline{\chi}_j\|_{L^p_\sigma(\Delta_{\rho_j}\cap\{e^{-N}<|z|<e^N\})}\\ &+\|(\overline{D}_{u\sharp_{\rho_i}v}-\overline{D}_{u\sharp_{\rho_j}v})\overline{\chi}_j\|_{L^p_\sigma(\Delta_{\rho_j}\cap\{e^{-N}<|z|<e^N\})}\\ \le\;&\|\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_i\|_{L^p_\sigma(\Delta_{\rho_i}\cap\{e^{-N}<|z|<e^N\})}+\|\overline{D}_{u\sharp_{\rho_j}v}\overline{\chi}_j\|_{L^p_\sigma(\Delta_{\rho_j}\cap\{e^{-N}<|z|<e^N\})}\\ &+C\|\overline{D}_{u\sharp_{\rho_i}v}-\overline{D}_{u\sharp_{\rho_j}v}\|_{C^0([-N,N]\times[0,\pi])}\|\overline{\chi}_j\|_{W^{1,p}_\sigma(\Delta_{\rho_j}\cap\{e^{-N}<|z|<e^N\})}\\ \to\;& 0,\end{aligned}$$

where $\|\overline{D}_{u\sharp_{\rho_i}v}-\overline{D}_{u\sharp_{\rho_j}v}\|_{C^0([-N,N]\times[0,\pi])}=\|a_i-a_j\|_{C^0([-N,N]\times[0,\pi])}+\|b_i-b_j\|_{C^0([-N,N]\times[0,\pi])}+\|c_i-c_j\|_{C^0([-N,N]\times[0,\pi])}$. Then we can conclude $\|e^{\sigma/p}_{\rho_i}\overline{\chi}_i-e^{\sigma/p}_{\rho_j}\overline{\chi}_j\|_{W^{1,p}([-N+1,N-1]\times[0,\pi])}\to 0$, and $\|\overline{\chi}_N-e^{\sigma/p}_{\rho_i}\overline{\chi}_i\|_{W^{1,p}([-N+1,N-1]\times[0,\pi])}\to 0$. Define $\overline{\chi}_\infty$ by $\overline{\chi}_\infty|_{[-N+1,N-1]\times[0,\pi]}=\overline{\chi}_N$. We remark that $\|\overline{\chi}_\infty\|_{W^{1,p}(\mathbf{R}\times[0,\pi])}<C$ from $\sup_{N,i}\|e^{\sigma/p}_{\rho_i}\overline{\chi}_i\|_{W^{1,p}([-N,N]\times[0,\pi])}<$

C. Moreover,

$$\begin{aligned}
& \|\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})\overline{\chi}_\infty+\frac{1}{2}\mathrm{sgn}(s)\frac{\sigma}{p}\overline{\chi}_\infty\|_{L^p([-N,N]\times[0,\pi])} \\
\leq\ & \|\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})(\overline{\chi}_\infty-e_{\rho_i}^{\sigma/p}\overline{\chi}_i)\|_{L^p([-N,N]\times[0,\pi])} \\
& +\|\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})e_{\rho_i}^{\sigma/p}\overline{\chi}_i+\frac{1}{2}\mathrm{sgn}(s)\frac{\sigma}{p}\overline{\chi}_\infty\|_{L^p([-N,N]\times[0,\pi])} \\
\leq\ & \|\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})(\overline{\chi}_\infty-e_{\rho_i}^{\sigma/p}\overline{\chi}_i)\|_{L^p([-N,N]\times[0,\pi])} \\
& +\|e_{\rho_i}^{\sigma/p}\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})\overline{\chi}_i\|_{L^p([-N,N]\times[0,\pi])} \\
& +\|-\frac{1}{2}\mathrm{sgn}(s)\frac{\sigma}{p}e_{\rho_i}^{\sigma/p}\overline{\chi}_i+\frac{1}{2}\mathrm{sgn}(s)\frac{\sigma}{p}\overline{\chi}_\infty\|_{L^p([-N,N]\times[0,\pi])} \\
\leq\ & C\|\overline{\chi}_\infty-e_{\rho_i}^{\sigma/p}\overline{\chi}_i\|_{W^{1,p}([-N,N]\times[0,\pi])} \\
& +\|e_{\rho_i}^{\sigma/p}\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})\overline{\chi}_i-e_{\rho_i}^{\sigma/p}\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_i\|_{L^p([-N,N]\times[0,\pi])} \\
& +\|e_{\rho_i}^{\sigma/p}\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_i\|_{L^p([-N,N]\times[0,\pi])}+C\|\overline{\chi}_\infty-e_{\rho_i}^{\sigma/p}\overline{\chi}_i\|_{L^p([-N,N]\times[0,\pi])} \\
\leq\ & C\|\overline{\chi}_\infty-e_{\rho_i}^{\sigma/p}\overline{\chi}_i\|_{W^{1,p}([-N,N]\times[0,\pi])} \\
& +\|\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}})-\overline{D}_{u\sharp_{\rho_i}v}\|_{C^0([-N,N]\times[0,\pi])}\|\overline{\chi}_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i}\cap\{e^{-N}<|z|<e^N\})} \\
& +\|\overline{D}_{u\sharp_{\rho_i}v}\overline{\chi}_i\|_{L^p_\sigma(\Delta_{\rho_i}\cap\{e^{-N}<|z|<e^N\})}+C\|\overline{\chi}_\infty-e_{\rho_i}^{\sigma/p}\overline{\chi}_i\|_{L^p([-N,N]\times[0,\pi])} \\
\to\ & 0.
\end{aligned}$$

Then we can conclude $\frac{1}{2}(\frac{\partial}{\partial s}+\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}}+\mathrm{sgn}(s)\frac{\sigma}{p})\overline{\chi}_\infty=0$ which is equivalent to the following equations:

$$\begin{aligned}
\frac{1}{2}(\frac{\partial}{\partial s}+\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial t}+\mathrm{sgn}(s)\frac{\sigma}{p})\overline{\chi}^1_\infty &= 0,\\
\frac{1}{2}(\frac{\partial}{\partial s}+J_0\frac{\partial}{\partial t}+\mathrm{sgn}(s)\frac{\sigma}{p})\overline{\chi}^2_\infty &= 0.
\end{aligned}$$

Put $\overline{\chi}^1_\infty=x\frac{\partial}{\partial\theta}+yX_\lambda$ and $z=x+iy$, and the first equation turns out to be $\frac{1}{2}(\frac{\partial}{\partial s}+i\frac{\partial}{\partial t}+\mathrm{sgn}(s)\frac{\sigma}{p})z=0$. By the separation of variables, we can solve this equation and $z=ce_0^{\sigma/p}$, for some $c\in\mathbf{R}$. Moreover, from the assumption $\chi_i^1(1)=0$, we have $\overline{\chi}^1_\infty=0$. Concerning the second equation, we get $\overline{\chi}^2_\infty=0$ from Lemma 6.2. Then $\|e_{\rho_i}^{\sigma/p}\overline{\chi}_i\|_{W^{1,p}([-3,3]\times[0,\pi])}\to 0$, which implies $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i}\cap\{e^{-3}<|z|<e^3\})}\to 0$. □

We define $\mathrm{Ker}D_u=\{\zeta\in T_uW^{1,p}_\sigma(\Theta;\gamma)|D_u\zeta=0\}$.

Lemma 6.4. *There exists some constant C depending only on u such that*

$$\|n\|_{W^{1,p}_\sigma(\Theta)}\leq C\|D_un\|_{L^p_\sigma(\Theta)}$$

for $n\in(KerD_u)^\perp=\{n\in T_uW^{1,p}_\sigma(\Theta;\gamma)|\int_\Theta\langle n,\zeta\rangle\alpha^{\sigma/p}(|z|)dxdy=0$ *for* $\zeta\in KerD_u\}$.

Proof. Suppose that there exists a sequence $\{n_i\}_{i=0}^\infty$ of $n_i\in(\mathrm{Ker}D_u)^\perp$ such that $\|n_i\|_{W^{1,p}_\sigma(\Theta)}=1$ and $\|D_un_i\|_{L^p_\sigma(\Theta)}\to 0$. Fix $N>1$. By the Rellich's theorem there exist a subsequence $\{n_{i_l}\}_{l=1}^{\infty}$ and $n_\infty \in L^p_\sigma(\Theta)$ such that $\|n_\infty - n_{i_l}\|_{L^p_\sigma(\Theta\cap\{|z|<e^N\})} \to 0$. We omit to mention subsequences hereafter. By the Gärding inequality

$$\begin{aligned} & \|n_i - n_j\|_{W^{1,p}_\sigma(\Theta\cap\{|z|<e^{N-1}\})} \\ \leq\ & C(\|D_u(n_i - n_j)\|_{L^p_\sigma(\Theta\cap\{|z|<e^N\})} + \|n_i - n_j\|_{L^p_\sigma(\Theta\cap\{|z|<e^N\})}) \\ \to\ & 0, \end{aligned}$$

and $\|n_\infty - n_i\|_{W^{1,p}_\sigma(\Theta\cap\{|z|<e^N\})} \to 0$. Moreover,

$$\|D_u n_\infty - D_u n_i\|_{L^p_\sigma(\Theta\cap\{|z|<e^N\})} \leq C\|n_\infty - n_i\|_{W^{1,p}_\sigma(\Theta\cap\{|z|<e^N\})} \to 0,$$

and $D_u n_\infty = 0$. Since $\|n_i\|_{W^{1,p}_\sigma(\Theta)} = 1$, $\|n_\infty\|_{W^{1,p}_\sigma(\Theta)} = 1$. So $n_\infty \in \mathrm{Ker}D_u$, which contradicts to $n_i \in (\mathrm{Ker}D_u)^\perp$. Hence there is no such sequence as $\{n_i\}_{i=1}^\infty$, and there exists some constant C as in the lemma. □

We define

$$V_\rho^\perp = \left\{ \chi \in T_{u\sharp_\rho v} W^{1,p}_\sigma(\Delta_\rho) \;\middle|\; \begin{array}{l} \int_{\Delta_\rho} \langle \chi, \zeta\sharp_\rho \eta\rangle \beta^{\sigma/p}_{\rho_i}(|z|)dxdy = 0 \text{ for } \zeta \in \mathrm{Ker}D_u \text{ and } \eta \in \mathrm{Ker}D_v \\ \text{and } \chi^1(1) = 0 \end{array} \right\}.$$

Since $\chi^1(1) \in \mathbf{R}(\frac{\partial}{\partial\theta})_{\overline{\gamma}(0)}$, the codimension of $V_\rho^\perp$ in $T_{u\sharp_\rho v}W^{1,p}_\sigma(\Delta_\rho)$ is equal to $\dim \mathrm{Ker}D_u + \dim \mathrm{Ker}D_v + 1$.

Proposition 6.5. *Let u and v be punctured pseudo-holomorphic discs. Then there exist some constants ρ_0 and C depending only on u and v such that*

$$\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} \leq C\|D_{u\sharp_\rho v}\chi\|_{L^p_\sigma(\Delta_\rho)},$$

for $\rho > \rho_0$ and $\chi \in V_\rho^\perp$.

Proof. Let $\{(\rho_i, \chi_i)\}_{i=1}^\infty$ be a sequence of pairs $\rho_i \in \mathbf{R}$ and $\chi_i \in V_{\rho_i}^\perp$. Suppose that $\rho_i \to \infty$ and that $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = 1$ and $\|D_{u\sharp_{\rho_i} v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$. Define smooth cutoff functions β_Θ, $\beta_{[-3,3]}$ and β_Ξ on Δ_ρ such that

$$\beta_\Theta(z) = \begin{cases} 1, & \text{for } |z| \leq e^{-3}, \\ 0, & \text{for } e^{-2} < |z|, \end{cases}$$

$$\beta_{[-3,3]}(z) = \begin{cases} 0, & \text{for } |z| < e^{-3}, \\ 1, & \text{for } e^{-2} < |z| < e^2, \\ 0, & \text{for } e^3 < |z|, \end{cases}$$

$$\beta_\Xi(z) = \begin{cases} 0, & \text{for } |z| < e^2, \\ 1, & \text{for } e^3 < |z|, \end{cases}$$

and $\beta_\Theta + \beta_{[-3,3]} + \beta_\Xi \equiv 1$. Then

$$\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} \leq \|\beta_\Theta\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} + \|\beta_{[-3,3]}\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} + \|\beta_\Xi\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})}.$$

From Proposition 6.3, $\|\beta_{[-3,3]}\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} \to 0$. Due to the support of $\beta_\Theta\chi_i$, we may think of $\beta_\Theta\chi_i \in T_{u\sharp_{\rho_i}v}W^{1,p}_\sigma(\Delta_{\rho_i})$ as $\beta_\Theta\chi_i \in T_u W^{1,p}_\sigma(\Theta)$. Let $\{e_1, \ldots, e_l\}$ be an orthonormal basis of $\mathrm{Ker}D_u$, i.e., $\int_\Theta \langle e_i, e_j\rangle \alpha^{\sigma/p}(|z|)dxdy = \delta_{ij}$. Decompose $\beta_\Theta\chi_i$ into $k_i + n_i$, where $k_i = \sum_{j=1}^l \int_\Theta \langle \beta_\Theta\chi_i, e_j\rangle \alpha^{\sigma/p}(|z|)dxdy e_j$ and $n_i = \beta_\Theta\chi_i - k_i$. Then $\|\beta_\Theta\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = \|\beta_\Theta\chi_i\|_{W^{1,p}_\sigma(\Theta)} \leq \|k_i\|_{W^{1,p}_\sigma(\Theta)} + \|n_i\|_{W^{1,p}_\sigma(\Theta)}$. By definition, for $e_j \in \mathrm{Ker}D_u$,

$$\int_{\Delta_{\rho_i}} \langle \chi_i, e_j \sharp_{\rho_i} 0\rangle \beta^{\sigma/p}_{\rho_i}(|z|)dxdy = 0.$$

And, due to the support of $e_j\sharp_{\rho_i}0$,

$$\begin{aligned}&\int_{\Delta_{\rho_i}}\langle\chi_i, e_j\sharp_{\rho_i}0\rangle\beta_{\rho_i}^{\sigma/p}(|z|)dxdy\\ =&\int_{\Theta}\langle\chi_i, e_j\sharp_{\rho_i}0\rangle\alpha^{\sigma/p}(|z|)dxdy\\ =&\int_{\Theta}\langle\beta_\Theta\chi_i, e_j\rangle\alpha^{\sigma/p}(|z|)dxdy+\int_{\Theta}(1-\beta_\Theta)\langle\chi_i, e_j\sharp_{\rho_i}0\rangle\alpha^{\sigma/p}(|z|)dxdy.\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\int_{\Theta}(1-\beta_\Theta)\langle\chi_i, e_j\sharp_{\rho_i}0\rangle\alpha^{\sigma/p}(|z|)dxdy\right| &\leq C\int_{\Theta\cap\{e^{\rho_i-3}<|z|<e^{\rho_i-1}\}}|\chi_i||e_j|\alpha^{\sigma/p}(|z|)dxdy\\ &\leq C\|\chi_i\|_{C^0(\Delta_{\rho_i})}\|e_j\|_{L^p_\sigma(\Theta\cap\{e^{\rho_i-3}<|z|<e^{\rho_i-1}\})}.\end{aligned}$$

Since $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})}=1$, we have $\|\chi_i\|_{C^0(\Delta_{\rho_i})}\leq C$. Hence $\int_\Theta\langle\beta_\Theta\chi_i, e_j\rangle\alpha^{\sigma/p}(|z|)dxdy\to 0$, and $\|k_i\|_{W^{1,p}_\sigma(\Theta)}\to 0$. From Proposition 6.3, Lemma 6.4 and

$$\begin{aligned}\|D_un_i\|_{L^p_\sigma(\Theta)} &= \|D_u(k_i+n_i)\|_{L^p_\sigma(\Theta)}\\ &= \|D_u(\beta_\Theta\chi_i)\|_{L^p_\sigma(\Theta)}\\ &= \|D_{u\sharp_{\rho_i}v}(\beta_\Theta\chi_i)\|_{L^p_\sigma(\Delta_{\rho_i})}\\ &\leq C(\|\chi_i\|_{L^p_\sigma(\Delta_{\rho_i}\cap\{e^{-3}<|z|<e^{-2})}+\|D_{u\sharp_{\rho_i}v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})}),\end{aligned}$$

we obtain $\|n_i\|_{W^{1,p}_\sigma(\Theta)}\to 0$. Hence $\|\beta_\Theta\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})}\to 0$. Similarly, we can prove $\|\beta_\Xi\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})}\to 0$, and finally we have $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})}\to 0$, which contradicts to $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})}=1$. Hence there is no such sequence as $\{(\rho_i,\chi_i)\}_{i=1}^\infty$, and there exists some constant C as in the proposition. □

Corollary 6.6. *Suppose that $D_u: T_uW^{1,p}_\sigma(\Theta;\gamma)\to L^p_\sigma(\Theta;\gamma)_u$ and $D_v: T_vW^{1,p}_\sigma(\Xi;\gamma)\to L^p_\sigma(\Xi;\gamma)_v$ are surjective. Then there are some constants ρ_0 and C depending only on u and v such that, for $\rho>\rho_0$, there exists $G_{u\sharp_\rho v}: L^p_\sigma(\Delta_\rho)_{u\sharp_\rho v}\to V_\rho^\perp$ which satisfies*

$$\begin{aligned}D_{u\sharp_\rho v}G_{u\sharp_\rho v} &= id,\\ \|G_{u\sharp_\rho v}\kappa\|_{W^{1,p}_\sigma(\Delta_\rho)} &\leq C\|\kappa\|_{L^p_\sigma(\Delta_\rho)}.\end{aligned}$$

Proof. From Proposition 6.5, if $\kappa\in\mathrm{Ker}D_{u\sharp_\rho v}\cap V_\rho^\perp$, then $\kappa=0$ and

$$\dim\mathrm{Ker}D_{u\sharp_\rho v}\leq\mathrm{codim}V_\rho^\perp=\dim\mathrm{Ker}D_u+\dim\mathrm{Ker}D_v+1.$$

We remark that, for small $\sigma>0$, the spectral flow tells us

$$\mathrm{Index}D_{u\sharp_\rho v}=\mathrm{Index}D_u+\dim\mathrm{Ker}\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}}+\mathrm{Index}D_v,$$

where Index means the Fredholm index. In fact $\dim\mathrm{Ker}\overline{\gamma}^*J\circ\overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}}=1$. Then the surjectivity of D_u and D_v implies that

$$\dim\mathrm{Ker}D_{u\sharp_\rho v}=\dim\mathrm{Ker}D_u+\dim\mathrm{Ker}D_v+1+\mathrm{Coker}D_{u\sharp_\rho v}.$$

Hence we obtain $\dim\mathrm{Ker}D_{u\sharp_\rho v}=\mathrm{codim}V_\rho^\perp$ and $\dim\mathrm{Coker}D_{u\sharp_\rho v}=0$, which imply that $\mathrm{Ker}D_{u\sharp_\rho v}\oplus V_\rho^\perp=T_{u\sharp_\rho v}W^{1,p}_\sigma(\Delta_\rho)$ and $D_{u\sharp_\rho v}: V_\rho^\perp\to L^p_\sigma(\Delta_\rho)_{u\sharp_\rho v}$ is surjective, and isomorphic. We define $G_{u\sharp_\rho v}$ by the inverse of $D_{u\sharp_\rho v}$, and the constant C as in the corollary is derived from the one of Proposition 6.5. □

We give Newton's method to find pseudo-holomorphic discs near to approximate pseudo-holomorphic discs [1] and [2].

Proposition 6.7. *For $w \in W^{1,p}_\sigma(\Delta_\rho)$, suppose that there exists some constant C which satisfies the following conditions:*

- $\|N_w(\chi)-N_w(\chi')\|_{L^p_\sigma(\Delta_\rho)} \leq C(\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)}+\|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)})\|\chi-\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)}$, *for $\chi, \chi' \in T_wW^{1,p}_\sigma(\Delta_\rho)$ with $\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)}, \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-2}/4$.*
- *There exists $G_w : L^p_\sigma(\Delta_\rho)_w \to T_wW^{1,p}_\sigma(\Delta_\rho)$ such that $D_wG_w = id$ and $\|G_w\kappa\|_{W^{1,p}_\sigma(\Delta_\rho)} \leq C\|\kappa\|_{L^p_\sigma(\Delta_\rho)}$.*
- $\|\mathcal{F}_w(0)\|_{L^p_\sigma(\Delta_\rho)} \leq C^{-3}/16$.

Then there exists $\chi \in T_wW^{1,p}_\sigma(\Delta_\rho)$ such that $\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} \leq C^{-2}/4$ and $\mathcal{F}_w(\chi) = 0$, which implies $\overline{\partial}_J(\exp_w^{g_{M\sharp_\rho M'}} \chi) = 0$.

Proof. For $\chi \in \mathrm{Ker}D_w$, we define $F_\chi : L^p_\sigma(\Delta_\rho)_w \to L^p_\sigma(\Delta_\rho)_w$ by

$$F_\chi(\kappa) = -\mathcal{F}_w(0) - N_w(\chi + G_w\kappa).$$

Put $\chi' = 0$ in the first condition, and $\|N_w(\chi)\|_{L^p_\sigma(\Delta_\rho)} \leq C\|\chi\|^2_{W^{1,p}_\sigma(\Delta_\rho)}$. Then

$$\begin{aligned} & \| - \mathcal{F}_w(0) - N_w(\chi + G_w\kappa)\|_{L^p_\sigma(\Delta_\rho)} \\ \leq\ & \|\mathcal{F}_w(0)\|_{L^p_\sigma(\Delta_\rho)} + \|N_w(\chi + G_w\kappa)\|_{L^p_\sigma(\Delta_\rho)} \\ \leq\ & \|\mathcal{F}_w(0)\|_{L^p_\sigma(\Delta_\rho)} + C\|\chi + G_w\kappa\|^2_{W^{1,p}_\sigma(\Delta_\rho)} \\ \leq\ & \|\mathcal{F}_w(0)\|_{L^p_\sigma(\Delta_\rho)} + C(\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} + C\|\kappa\|_{L^p_\sigma(\Delta_\rho)})^2. \end{aligned}$$

For $x, y \in L^p_\sigma(\Delta_\rho)_w$,

$$\begin{aligned} & \| - N_w(\chi + G_wx) + N_w(\chi + G_wy)\|_{L^p_\sigma(\Delta_\rho)} \\ \leq\ & C(\|\chi + G_wx\|_{W^{1,p}_\sigma(\Delta_\rho)} + \|\chi + G_wy\|_{W^{1,p}_\sigma(\Delta_\rho)})\|G_wx - G_wy\|_{W^{1,p}_\sigma(\Delta_\rho)} \\ \leq\ & C^2(2\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} + C\|x\|_{L^p_\sigma(\Delta_\rho)} + C\|y\|_{L^p_\sigma(\Delta_\rho)})\|x - y\|_{W^{1,p}_\sigma(\Delta_\rho)}. \end{aligned}$$

Define $B_\chi = \{\chi \in \mathrm{Ker}D_w | \|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-2}/8\}$ and $B_\kappa = \{\kappa \in L^p_\sigma(\Delta_\rho)_w | \|\kappa\|_{L^p_\sigma(\Delta_\rho)} < C^{-3}/8\}$. Then, if $\chi \in B_\chi$, $F_\chi : B_\kappa \to B_\kappa$ and

$$\|F_\chi(x) - F_\chi(y)\|_{L^p_\sigma(\Delta_\rho)} \leq \frac{1}{2}\|x - y\|_{L^p_\sigma(\Delta_\rho)},$$

for $x, y \in B_\kappa$. By the contraction theorem, for each $\chi \in B_\chi$, we can find κ_χ such that $F_\chi(\kappa_\chi) = \kappa_\chi$ which implies

$$-\mathcal{F}_w(0) - N_w(\chi + G_w\kappa_\chi) = \kappa_\chi.$$

Define $f(\chi) = G_w\kappa_\chi$, and

$$\mathcal{F}_w(0) + D_w(\chi + f(\chi)) + N_w(\chi + f(\chi)) = 0$$

since $\chi \in \mathrm{Ker}D_u$ and $D_wG_w = \mathrm{id}$. This implies

$$\overline{\partial}_J(\exp_w^{g_{M\sharp_\rho M'}}(\chi + f(\chi))) = 0,$$

for $\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-2}/8$. And $\|\chi + f(\chi)\|_{W^{1,p}_\sigma(\Delta_\rho)} \leq C^{-2}/8 + CC^{-3}/8 = C^{-2}/4$. □

Finally, we glue the punctured pseudo-holomorphic discs u and v. From Lemma 5.1, there is ρ_1 such that, for $\rho > \rho_1$, there exists some constant C_1 which satisfies the first condition of Proposition 6.7. Similarly, from Corollary 6.6, we have ρ_2 such that, for $\rho > \rho_2$, there exists some constant C_2 and the second condition of Proposition 6.7 holds. And, from Lemma 6.1, there is ρ_3 such that, for $\rho > \rho_3$, there exists some constant C_3 which satisfies the third condition of Proposition 6.7. Put $\rho_0 = \max(\rho_1, \rho_2, \rho_3)$ and $C = \max(C_1, C_2, C_3)$, and we can apply Proposition 6.7 to our $w = u\sharp_\rho v$, for $\rho > \rho_0$, and get a pseudo-holomorphic disc near to w.

7. Degenerate Reeb chords

In this section, we discuss the gluing constructions of pseudo-holomorphic discs with degenerate Reeb chords, i.e., we do not assume that γ is nondegenerate. We can use Lemma 5.1, Lemma 6.1, Lemma 6.4 and Proposition 6.7, where we do not need the nondegeneracy.

Let d be the dimension of $T_{\overline{\gamma}(0)}\Lambda \cap (d\varphi_T)^{-1}T_{\overline{\gamma}(\pi)}\Lambda$. We may choose e_i as in Section 5 such that $\{e_1, \dots, e_d\}$ is a basis of $T_{\overline{\gamma}(0)}\Lambda \cap (d\varphi_T)^{-1}T_{\overline{\gamma}(\pi)}\Lambda$ and $\{e_1, \dots, e_n\}$ is a basis of $T_{\overline{\gamma}(0)}\Lambda$. Then, if $\overline{\gamma}^*J \circ \overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}}\zeta(t) = 0$ with $\zeta(0) \in \mathbf{R}\frac{\partial}{\partial\theta} \oplus T_{\overline{\gamma}(0)}\Lambda$ and $\zeta(\pi) \in \mathbf{R}\frac{\partial}{\partial\theta} \oplus T_{\overline{\gamma}(\pi)}\Lambda$, we have $\zeta(t) = c\frac{\partial}{\partial\theta} \oplus \sum_{i=1}^d c_i e_i(t)$, for $c, c_i \in \mathbf{R}$.

Suppose that $(d\varphi_T)^{-1}T_{\overline{\gamma}(\pi)}\Lambda$ is spanned by $\{e_1, \dots, e_d, f_{d+1}, \dots, f_n\}$, where $f_i \in \bigoplus_{i=d+1}^n(\mathbf{R}e_i \oplus \mathbf{R}J_\xi e_i)$. Let $\Lambda_0 \subset \mathbf{R}^{2(n-d)}$ be the $(n-d)$-dimensional linear subspace corresponding to $\bigoplus_{i=d+1}^n \mathbf{R}e_i \subset \bigoplus_{i=d+1}^n \mathbf{R}e_i \oplus \mathbf{R}J_\xi e_i$ and $\Lambda_\pi \subset \mathbf{R}^{2(n-d)}$ the $(n-d)$-dimensional linear subspace corresponding to $\bigoplus_{i=d+1}^n \mathbf{R}f_i \subset \bigoplus_{i=d+1}^n \mathbf{R}e_i \oplus \mathbf{R}J_\xi e_i$. We remark that Λ_0 and Λ_π intersect transversely in $\mathbf{R}^{2(n-d)}$. Moreover, we define

$$W^{1,p}(\mathbf{R}\times[0,\pi], \mathbf{R}^{2(n-d)}, \Lambda_0, \Lambda_\pi) = \{\chi \in W^{1,p}(\mathbf{R}\times[0,\pi], \mathbf{R}^{2(n-d)}) | \chi(0) \in \Lambda_0 \text{ and } \chi(\pi) \in \Lambda_\pi\}$$

and

$$W^{1,p}([0,\pi], \mathbf{R}^{2(n-d)}, \Lambda_0, \Lambda_\pi) = \{\chi \in W^{1,p}([0,\pi], \mathbf{R}^{2(n-d)}) | \chi(0) \in \Lambda_0 \text{ and } \chi(\pi) \in \Lambda_\pi\},$$

and obtain the following lemma in a completely similar way to Lemma 6.2.

Lemma 7.1. *If $\sigma > 0$ is small enough, the operator $\frac{\partial}{\partial s} + J_0\frac{\partial}{\partial t} + sgn(s)\frac{\sigma}{p} : W^{1,p}(\mathbf{R} \times [0,\pi], \mathbf{R}^{2(n-d)}, \Lambda_0, \Lambda_\pi) \to L^p(\mathbf{R} \times [0,\pi], \mathbf{R}^{2(n-d)})$ is bijective, for $1 < p < \infty$.*

For $\chi \in T_{\overline{\gamma}(0)}(\mathbf{R}\times N)$, we denote by χ^1 the $\mathbf{R}\frac{\partial}{\partial\theta} \oplus \mathbf{R}X_\lambda \oplus \bigoplus_{i=1}^d \mathbf{R}e_i(0) \oplus \mathbf{R}J_\xi e_i(0)$ component of χ and by χ^2 the $\bigoplus_{i=d+1}^n \mathbf{R}e_i(0) \oplus \mathbf{R}J_\xi e_i(0)$ component of χ, and obtain the following lemma in a completely similar way to Lemma 6.3.

Proposition 7.2. *Let u and v be punctured pseudo-holomorphic discs and $\{(\rho_i, \chi_i)\}_{i=1}^\infty$ a sequence of pairs $\rho_i \in \mathbf{R}$ and $\chi_i \in T_{u\sharp_{\rho_i}v}W^{1,p}_\sigma(\Delta_{\rho_i})$. Suppose that $\rho_i \to \infty$ and that $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = 1$, $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$ and $\chi_i^1(1) = 0$. Then there exists a subsequence $\{(\rho_{i_l}, \chi_{i_l})\}_{l=1}^\infty$ such that*

$$\|\chi_{i_l}\|_{W^{1,p}_\sigma(\Delta_{\rho_{i_l}} \cap \{e^{-3} < |z| < e^3\})} \to 0.$$

We define

$$V_\rho^\perp = \left\{ \chi \in T_{u\sharp_\rho v}W^{1,p}_\sigma(\Delta_\rho) \;\middle|\; \begin{array}{l} \int_{\Delta_\rho} \langle \chi, \zeta\sharp_\rho\eta\rangle \beta_{\rho_i}^{\sigma/p}(|z|)dxdy = 0 \text{ for } \zeta \in \mathrm{Ker}D_u \text{ and } \eta \in \mathrm{Ker}D_v \\ \text{and } \chi^1(1) = 0 \end{array} \right\}.$$

Since $\chi^1(1) \in \mathbf{R}(\frac{\partial}{\partial\theta})_{\overline{\gamma}(0)} \oplus \bigoplus_{i=1}^d \mathbf{R}e_i(0)$, the codimension of $V_\rho^\perp$ in $T_{u\sharp_\rho v}W_\sigma^{1,p}(\Delta_\rho)$ is equal to $\dim \mathrm{Ker}D_u + \dim \mathrm{Ker}D_v + d + 1$. Then we obtain the following proposition in a completely similar way to Proposition 6.5.

Proposition 7.3. *Let u and v be punctured pseudo-holomorphic discs. Then there exist some constants ρ_0 and C depending only on u and v such that*

$$\|\chi\|_{W_\sigma^{1,p}(\Delta_\rho)} \le C\|D_{u\sharp_\rho v}\chi\|_{L_\sigma^p(\Delta_\rho)}$$

for $\rho > \rho_0$ and $\chi \in V_\rho^\perp$.

Corollary 7.4. *Suppose that $D_u : T_uW_\sigma^{1,p}(\Theta;\gamma) \to L_\sigma^p(\Theta;\gamma)_u$ and $D_v : T_vW_\sigma^{1,p}(\Xi;\gamma) \to L_\sigma^p(\Xi;\gamma)_v$ are surjective. Then there are some constants ρ_0 and C depending only on u and v such that, for $\rho > \rho_0$, there exists $G_{u\sharp_\rho v} : L_\sigma^p(\Delta_\rho)_{u\sharp_\rho v} \to V_\rho^\perp$ which satisfies*

$$\begin{aligned} D_{u\sharp_\rho v}G_{u\sharp_\rho v} &= id, \\ \|G_{u\sharp_\rho v}\kappa\|_{W_\sigma^{1,p}(\Delta_\rho)} &\le C\|\kappa\|_{L_\sigma^p(\Delta_\rho)}. \end{aligned}$$

Proof. From Proposition 7.3, if $\kappa \in \mathrm{Ker}D_{u\sharp_\rho v} \cap V_\rho^\perp$, then $\kappa = 0$ and

$$\dim \mathrm{Ker}D_{u\sharp_\rho v} \le \mathrm{codim}V_\rho^\perp = \dim \mathrm{Ker}D_u + \dim \mathrm{Ker}D_v + d + 1.$$

We remark that, for small $\sigma > 0$, the spectral flow tells us

$$\mathrm{Index}D_{u\sharp_\rho v} = \mathrm{Index}D_u + \dim \mathrm{Ker}\overline{\gamma}^*J \circ \overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}} + \mathrm{Index}D_v,$$

where Index means the Fredholm index. In fact $\dim \mathrm{Ker}\overline{\gamma}^*J \circ \overline{\gamma}^*\nabla_{\frac{\partial}{\partial t}} = d+1$. Then the surjectivity of D_u and D_v implies that

$$\dim \mathrm{Ker}D_{u\sharp_\rho v} = \dim \mathrm{Ker}D_u + \dim \mathrm{Ker}D_v + d + 1 + \mathrm{Coker}D_{u\sharp_\rho v}.$$

Hence we obtain $\dim \mathrm{Ker}D_{u\sharp_\rho v} = \mathrm{codim}V_\rho^\perp$ and $\dim \mathrm{Coker}D_{u\sharp_\rho v} = 0$, which imply that $\mathrm{Ker}D_{u\sharp_\rho v} \oplus V_\rho^\perp = T_{u\sharp_\rho v}W_\sigma^{1,p}(\Delta_\rho)$ and $D_{u\sharp_\rho v} : V_\rho^\perp \to L_\sigma^p(\Delta_\rho)_{u\sharp_\rho v}$ is surjective, and isomorphic. We define $G_{u\sharp_\rho v}$ by the inverse of $D_{u\sharp_\rho v}$, and the constant C as in the corollary is derived from the one of Proposition 7.3. □

Finally, we glue the punctured pseudo-holomorphic discs u and v. From Lemma 5.1, there is ρ_1 such that, for $\rho > \rho_1$, there exists some constant C_1 which satisfies the first condition of Proposition 6.7. Similarly, from Corollary 7.4, we have ρ_2 such that, for $\rho > \rho_2$, there exists some constant C_2 and the second condition of Proposition 6.7 holds. And, from Lemma 6.1, there is ρ_3 such that, for $\rho > \rho_3$, there exists some constant C_3 which satisfies the third condition of Proposition 6.7. Put $\rho_0 = \max(\rho_1, \rho_2, \rho_3)$ and $C = \max(C_1, C_2, C_3)$, and we can apply Proposition 6.7 to our $w = u\sharp_\rho v$, for $\rho > \rho_0$, and get a pseudo-holomorphic disc near to w.

8. Non-surjective Cauchy-Riemann operators

In this section, we discuss the gluing constructions of Kuranishi maps as in [3] with non-surjective linearized Cauchy-Riemann operators, i.e., we do not assume that D_u and D_v are surjective.

For $u \in W_\sigma^{1,p}(\Theta;\gamma)$, $\mathrm{Im}D_u \subset L_\sigma^p(\Theta;\gamma)_u$ is closed and $d_u = \dim L_\sigma^p(\Theta;\gamma)_u/\mathrm{Im}D_u$ is finite. We define $E_u \subset L_\sigma^p(\Theta;\gamma)_u$ by a d_u-dimensional linear subspace such that $\mathrm{Im}D_u + E_u = L_\sigma^p(\Theta;\gamma)_u$. Let $\{e_1^u, \ldots, e_{d_u}^u\}$ be a basis of E_u. Similarly, for $v \in W_\sigma^{1,p}(\Xi;\gamma)$, $\mathrm{Im}D_v \subset L_\sigma^p(\Xi;\gamma)_v$ is closed and $d_v = \dim L_\sigma^p(\Xi;\gamma)_v/\mathrm{Im}D_v$ is finite. We define $E_u \subset L^p_\sigma(\Theta;\gamma)_u$ by a d_v-dimensional linear subspace such that $\mathrm{Im}D_v + E_v = L^p_\sigma(\Xi;\gamma)_v$. Let $\{e^v_1,\dots,e^v_{d_v}\}$ be a basis of E_v.

For $a \in E_u$ and $b \in E_v$, we define $a\sharp_\rho b \in L^p_\sigma(\Delta_\rho)_{u\sharp_\rho v}$ by

$$a\sharp_\rho b = \begin{cases} a(e^\rho z), & \text{for } |z| \le e^{-3}, \\ \beta_u(\log|z|+2)a(e^\rho z), & \text{for } e^{-3} < |z| \le 1, \\ \beta_v(\log|z|-2)b(e^{-\rho} z), & \text{for } 1 < |z| \le e^3, \\ b(e^{-\rho} z), & \text{for } |z| > e^3, \end{cases}$$

and $E_{u\sharp_\rho v} = \{a\sharp_\rho b | a \in E_u \text{ and } b \in E_v\} \subset L^p_\sigma(\Delta_\rho)_{u\sharp_\rho v}$. Since the norm on the quotient $\overline{L}^p_\sigma(\Delta_\rho)_{u\sharp_\rho v} = L^p_\sigma(\Delta_\rho)_{u\sharp_\rho v}/E_{u\sharp_\rho v}$ is given by $\|\cdot\|_{\overline{L}^p_\sigma(\Delta_\rho)} = \inf_{k\in E_{u\sharp_\rho v}}\|\cdot + k\|_{L^p_\sigma(\Delta_\rho)}$, we obtain $\|\cdot\|_{\overline{L}^p_\sigma(\Delta_\rho)} \le \|\cdot\|_{L^p_\sigma(\Delta_\rho)}$, and slight modifications of Lemma 5.1 and Lemma 6.1, where we do not need the surjectivity, hold.

Lemma 8.1. *For $w \in W^{1,p}_\sigma(\Delta_\rho)$, we write $\mathcal{F}_w(\chi) = \mathcal{F}_w(0) + D_w\chi + N_w(\chi)$. Then there exists some constant C depending only on $\|\nabla w\|_{L^p(\Delta_\rho)}$ such that*

$$\|N_w(\chi) - N_w(\chi')\|_{\overline{L}^p_\sigma(\Delta_\rho)} \le C(\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} + \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)})\|\chi-\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)},$$

for $\chi,\chi' \in T_wW^{1,p}_\sigma(\Delta_\rho)$ with $\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)}, \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-1}$.

Lemma 8.2. *Let u and v be punctured pseudo-holomorphic discs. For any $\varepsilon > 0$, there exists some constant ρ_0 depending only on ε, u and v such that*

$$\|\overline{\partial}_J(u\sharp_\rho v)\|_{\overline{L}^p_\sigma(\Delta_\rho)} < \varepsilon,$$

for $\rho > \rho_0$.

Now we prove the new lemma.

Lemma 8.3. *Let u and v be punctured pseudo-holomorphic discs and $\{(\rho_i,\chi_i)\}_{i=1}^\infty$ a sequence of pairs $\rho_i \in \mathbf{R}$ and $\chi_i \in T_{u\sharp_{\rho_i}v}W^{1,p}_\sigma(\Delta_{\rho_i})$. Suppose that $\rho_i \to \infty$ and that $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = 1$, $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{\overline{L}^p_\sigma(\Delta_{\rho_i})} \to 0$. Then $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$.*

Proof. From $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{\overline{L}^p_\sigma(\Delta_{\rho_i})} \to 0$, there exists a sequence of $k_i \in E_{u\sharp_{\rho_i}v}$ such that $\|D_{u\sharp_{\rho_i}v}\chi_i + k_i\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$. And from $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = 1$, we have $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})} < C$. Hence we may think that $\|k_i\|_{L^p_\sigma(\Delta_{\rho_i})} < 2C$. Put

$$k_i = \sum_{p=1}^{d_u} c^u_{pi} e^u_p \sharp_{\rho_i} 0 + \sum_{q=1}^{d_v} c^v_{qi} 0 \sharp_{\rho_i} e^v_q,$$

for $c^u_{pi}, c^v_{qi} \in \mathbf{R}$. Because $\|k_i\|_{L^p_\sigma(\Delta_{\rho_i})} < 2C$, there exist c^u_p and c^v_q such that $\lim_{i\to\infty} c^u_{pi} = c^u_p$ and $\lim_{i\to\infty} c^v_{qi} = c^v_q$ after taking subsequences if necessary. Then we put

$$k'_i = \sum_{p=1}^{d_u} c^u_p e^u_p \sharp_{\rho_i} 0 + \sum_{q=1}^{d_v} c^v_q 0 \sharp_{\rho_i} e^v_q,$$

and $\|D_{u\sharp_{\rho_i}v}\chi_i + k_i'\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$. Moreover, due to the support of the elements of $E_{u\sharp_{\rho_i}v}$, we have

$$\begin{aligned} & \|D_{u\sharp_{\rho_i}v}\chi_i + k_i'\|_{L^p_\sigma(\Delta_\rho)} \\ = \; & \|D_u(\beta_u(\log|z| - \rho_i + 1)\chi_i(e^{-\rho_i}z)) + \sum_{p=1}^{d_u} c^u_p \beta_u(\log|z| - \rho_i + 2)e^u_p\|_{L^p_\sigma(\Theta)} \\ & + \|D_v(\beta_v(\log|z| + \rho_i - 1)\chi_i(e^{\rho_i}z)) + \sum_{q=1}^{d_v} c^v_q \beta_v(\log|z| + \rho_i - 2)e^v_q\|_{L^p_\sigma(\Xi)}. \end{aligned}$$

And there is some constant $C > 0$ such that

$$\begin{aligned} & \|D_u(\beta_u(\log|z| - \rho_i + 1)\chi_i(e^{-\rho_i}z)) + \sum_{p=1}^{d_u} c^u_p \beta_u(\log|z| - \rho_i + 2)e^u_p\|_{L^p_\sigma(\Theta)} \\ \geq \; & \|D_u(\beta_u(\log|z| - \rho_i + 1)\chi_i(e^{-\rho_i}z)) + \sum_{p=1}^{d_u} c^u_p e^u_p\|_{L^p_\sigma(\Theta)} - C\sum_{p=1}^{d_u} \|e^u_p\|_{L^p_\sigma(\Theta \cap \{e^{\rho_i - 3} < |z|\})}. \end{aligned}$$

Hence $D_u(\beta_u(\log|z| - \rho_i + 1)\chi_i(e^{-\rho_i}z)) \in \mathrm{Im}D_u$ converges to $\sum_{p=1}^{d_u} c^u_p e^u_p \in E_u$, and the limit is equal to 0 and $c^u_p = 0$. Similarly we obtain $c^v_q = 0$. Hence $k_i' = 0$ and $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$. □

From Lemma 8.3 and Proposition 7.2, we obtain the following proposition.

Proposition 8.4. *Let u and v be punctured pseudo-holomorphic discs and $\{(\rho_i, \chi_i)\}_{i=1}^\infty$ a sequence of pairs $\rho_i \in \mathbf{R}$ and $\chi_i \in T_{u\sharp_{\rho_i}v}W^{1,p}_\sigma(\Delta_{\rho_i})$. Suppose that $\rho_i \to \infty$ and that $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = 1$, $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{\overline{L}^p_\sigma(\Delta_{\rho_i})} \to 0$ and $\chi_i^1(1) = 0$. Then there exists a subsequence $\{(\rho_{i_l}, \chi_{i_l})\}_{l=1}^\infty$ such that*

$$\|\chi_{i_l}\|_{W^{1,p}_\sigma(\Delta_{\rho_{i_l}} \cap \{e^{-3} < |z| < e^3\})} \to 0.$$

And similarly, from Lemma 8.3, we obtain the following proposition which is a slight modification of Proposition 7.3.

Proposition 8.5. *Let u and v be punctured pseudo-holomorphic discs. Then there exist some constants ρ_0 and C depending only on u and v such that*

$$\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} \leq C\|D_{u\sharp_\rho v}\chi\|_{\overline{L}^p_\sigma(\Delta_\rho)}$$

for $\rho > \rho_0$ and $\chi \in V_\rho^\perp$.

Proof. Let $\{(\rho_i, \chi_i)\}_{i=1}^\infty$ be a sequence of pairs $\rho_i \in \mathbf{R}$ and $\chi_i \in V_{\rho_i}^\perp$. Suppose that $\rho_i \to \infty$ and that $\|\chi_i\|_{W^{1,p}_\sigma(\Delta_{\rho_i})} = 1$ and $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{\overline{L}^p_\sigma(\Delta_{\rho_i})} \to 0$. From Lemma 8.3, $\|D_{u\sharp_{\rho_i}v}\chi_i\|_{L^p_\sigma(\Delta_{\rho_i})} \to 0$. Then we obtain the same contradiction in the proof of Proposition 6.5, and hence there is no such sequence as $\{(\rho_i, \chi_i)\}_{i=1}^\infty$, and there exists some constant C as in the proposition. □

Corollary 8.6. *There are some constants ρ_0 and C depending only on u and v such that, for $\rho > \rho_0$, there exists $\overline{G}_{u\sharp_\rho v} : \overline{L}^p_\sigma(\Delta_\rho)_{u\sharp_\rho v} \to V_\rho^\perp$ which satisfies*

$$\begin{aligned} D_{u\sharp_\rho v}\overline{G}_{u\sharp_\rho v} &= id, \\ \|\overline{G}_{u\sharp_\rho v}\kappa\|_{W^{1,p}_\sigma(\Delta_\rho)} &\leq C\|\kappa\|_{\overline{L}^p_\sigma(\Delta_\rho)}. \end{aligned}$$

Proof. Here we denote by $\overline{D}_{u\sharp_\rho v}$ the composition of $D_{u\sharp_\rho v}$ and the projection $L^p_\sigma(\Delta_\rho)_{u\sharp_\rho v} \to \overline{L}^p_\sigma(\Delta_\rho)_{u\sharp_\rho v}$. From Proposition 8.5, if $\kappa \in \mathrm{Ker}\overline{D}_{u\sharp_\rho v} \cap V_\rho^\perp$, then $\kappa = 0$ and

$$\dim \mathrm{Ker}\overline{D}_{u\sharp_\rho v} \leq \mathrm{codim} V_\rho^\perp = \dim \mathrm{Ker} D_u + \dim \mathrm{Ker} D_v + d + 1.$$

We remark that, for small $\sigma > 0$, the spectral flow tells us

$$\mathrm{Index} D_{u\sharp_\rho v} = \mathrm{Index} D_u + \dim \mathrm{Ker}\overline{\gamma}^* J \circ \overline{\gamma}^* \nabla_{\frac{\partial}{\partial t}} + \mathrm{Index} D_v.$$

In fact $\dim \mathrm{Ker}\overline{\gamma}^* J \circ \overline{\gamma}^* \nabla_{\frac{\partial}{\partial t}} = d+1$ and $\dim E_{u\sharp_\rho v} = \dim \mathrm{Coker} D_u + \dim \mathrm{Coker} D_v$. Then

$$\dim \mathrm{Ker} D_{u\sharp_\rho v} = \dim \mathrm{Ker} D_u + \dim \mathrm{Ker} D_v + d + 1 + \dim \mathrm{Coker} D_{u\sharp_{rho} v} - \dim E_{u\sharp_\rho v}.$$

Since $\dim E_{u\sharp_\rho u} - (\dim \mathrm{Ker}\overline{D}_{u\sharp_\rho v} - \dim \mathrm{Ker} D_{u\sharp_\rho v}) \leq \dim \mathrm{Coker} D_{u\sharp_\rho v}$, we obtain

$$\dim \mathrm{Ker}\overline{D}_{u\sharp_\rho v} = \mathrm{codim} V_\rho^\perp$$

and

$$\dim \mathrm{Ker}\overline{D}_{u\sharp_\rho v} - \dim \mathrm{Ker} D_{u\sharp_\rho v} = \dim E_{u\sharp_\rho v} - \dim \mathrm{Coker} D_{u\sharp_\rho v}$$

which imply that $\mathrm{Ker}\overline{D}_{u\sharp_\rho v} \oplus V_\rho^\perp = T_{u\sharp_\rho v} W^{1,p}_\sigma(\Delta_\rho)$ and $\overline{D}_{u\sharp_\rho v} : V_\rho^\perp \to \overline{L}^p_\sigma(\Delta_\rho)_{u\sharp_\rho v}$ is surjective, and isomorphic. We can define $\overline{G}_{u\sharp_\rho v}$ by the inverse of $\overline{D}_{u\sharp_\rho v}$, and the constant C as in the corollary is derived from the one of Proposition 8.5. □

We give Newton's method to construct a Kuranishi map. The proof is completely similar to that of Proposition 6.7.

Proposition 8.7. *For $w \in W^{1,p}_\sigma(\Delta_\rho)$, let $E_w \subset L^p_\sigma(\Delta)_w$ be a finite dimensional linear subspace and $\overline{L}^p_\sigma(\Delta)_w = L^p_\sigma(\Delta)_w / E_w$. Suppose that there exists some constant C which satisfies the following conditions:*

- $\|N_w(\chi) - N_w(\chi')\|_{\overline{L}^p_\sigma(\Delta_\rho)} \leq C(\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} + \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)})\|\chi - \chi'\|_{W^{1,p}_\sigma(\Delta_\rho)}$, *for* $\chi, \chi' \in T_w W^{1,p}_\sigma(\Delta_\rho)$ *with* $\|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)}, \|\chi'\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-2}/4$.
- *There exists* $\overline{G}_w : \overline{L}^p_\sigma(\Delta_\rho)_w \to T_w W^{1,p}_\sigma(\Delta_\rho)$ *such that* $D_w \overline{G}_w = id$ *and* $\|\overline{G}_w \kappa\|_{W^{1,p}_\sigma(\Delta_\rho)} \leq C\|\kappa\|_{\overline{L}^p_\sigma(\Delta_\rho)}$.
- $\|\mathcal{F}_w(0)\|_{\overline{L}^p_\sigma(\Delta_\rho)} \leq C^{-3}/16$.

Then there exists a map $f : \{\chi \in KerD_w | \|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-2}/8\} \to V_\rho^\perp$ such that $\mathcal{F}_w(\chi + f(\chi)) = 0 \in \overline{L}^p_\sigma(\Delta)_w$ which implies $\overline{\partial}_J(\exp_w^{g_{M\sharp_\rho M'}}(\chi + f(\chi))) \in E_w$.

Finally, we construct the Kuranishi map. From Lemma 8.1, there is ρ_1 such that, for $\rho > \rho_1$, there exists some constant C_1 which satisfies the first condition of Proposition 8.7. Similarly, from Corollary 8.6, we have ρ_2 such that, for $\rho > \rho_2$, there exists some constant C_2 and the second condition of Proposition 8.7 holds. And, from Lemma 8.2, there is ρ_3 such that, for $\rho > \rho_3$, there exists some constant C_3 which satisfies the third condition of Proposition 8.7. Put $\rho_0 = \max(\rho_1, \rho_2, \rho_3)$ and $C = \max(C_1, C_2, C_3)$, and we can apply Proposition 8.7 to our $w = u\sharp_\rho v$, for $\rho > \rho_0$, and get the Kuranishi map $s_w(\chi) = \overline{\partial}_J(\exp_w^{g_{M\sharp_\rho M'}}(\chi + f(\chi))) \in E_w$ on $\{\chi \in \mathrm{Ker} D_w | \|\chi\|_{W^{1,p}_\sigma(\Delta_\rho)} < C^{-2}/8\}$.

We remark that, if $s_w(\chi) = 0$, then $\exp_w^{g_{M\sharp_\rho M'}}(\chi + f(\chi))$ is a pseudo-holomorphic disc near to w.

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Department of Mathematics, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan

E-mail address: akaho@comp.metro-u.ac.jp