CUP PRODUCTS ON MORSE HOMOLOGY OF MANIFOLDS WITH BOUNDARY

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ABSTRACT. We describe cup products on Morse homology of manifolds with boundary. As an application, we define product structures on Floer homology of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end. In particular, we show that these products satisfy the Leibniz rules on the chain level.

1. Introduction

In this paper we describe cup products on Morse homology of manifolds with boundary. As an application, we define product structures on Floer homology of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end. In particular, we show that these products satisfy the Leibniz rules on the chain level.

In [9] Witten invented Morse complex; For a Morse function on a closed manifold, the complex is generated by the critical points, and the boundary operator counts gradient trajectories between critical points of Morse index difference 1. The homology of Morse complex is called Morse homology, and it is isomorphic to the singular homology, see [4], [9] and Section 2. In [1] the author introduced Morse complex of manifolds *with boundary*; For some Morse function on a compact manifold with boundary, the complex is generated by the *interior* critical points and the *positive boundary* critical points, and the boundary operator counts *broken* gradient trajectories between generators of Morse index difference 1, and the homology is isomorphic to the absolute singular homology, see [1] and Section 2. As an application, the author introduced Floer homology for pairs of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end, see [1] and Section 5.

Although a single Morse function tells us the singular homology, Fukaya found that we need three Morse functions to describe cup products in terms of Morse theory, see [5] and Section 3; The cup products count gradient trees and satisfy the Leibniz rules on the chain level. (In fact Fukaya invented A_∞ structures among smooth functions on a closed manifold, see [5].) In this paper we describe cup products on Morse homology of manifolds with boundary, which also satisfy the Leibniz rules on the chain level, see Section 3 and Section 4. As an application, we define product structures on Floer homology of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end, see Section 5.

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We confirm our contents: In Section 2 we review Morse homology of compact manifolds with and without boundary. We emphasize the importance of unstable manifolds of Morse functions to understand Morse complex. In Section 3, we deal with cup products on Morse complex of compact manifolds with and without boundary: First we describe the cup product in terms of unstable manifolds, and secondly we *heuristically* obtain the cup products in terms of gradient trees. In particular, we prove the Leibniz rules in terms of unstable manifolds in Section 3. In Section 4, we again review Morse complex of manifolds with boundary, and prove the Leibniz rules on Morse complex of manifolds with boundary in terms of gradient trees. Finally, in Section 5, we review Floer homology of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end, and give product structures on the Floer homology, which satisfy the Leibniz rules on the chain level.

2. Morse homology of manifolds with boundary

In this section, we briefly review Morse homology of manifolds with boundary, introduced in [1]. But, before manifolds with boundary, we recall Morse homology of closed manifolds, see also [4] and [9].

Let M be an n-dimensional oriented closed manifold, and g a Riemannian metric on M. Let f be a Morse function on M. We denote by X_f the gradient vector field on M with respect to f and g, i.e., X_f is given by $df = g(X_f, \cdot)$. Let $\varphi_t : M \to M$ be the isotopy of $-X_f$, i.e., φ_t satisfy $d\varphi_t/dt = -X_f \circ \varphi_t$ and $\varphi_0(x) = x$. Then, for a critical point p of f, we define the stable manifold S_p by

$$S_p := \left\{ x \in M : \lim_{t \to +\infty} \varphi_t(x) = p \right\},$$

and similarly, the unstable manifold U_p by

$$U_p := \left\{ x \in M : \lim_{t \to -\infty} \varphi(x) = p \right\}.$$

Note that S_p is diffeomorphic to the $(n - \mu(p))$-dimensional open ball, and U_p is diffeomorphic to the $\mu(p)$-dimensional open ball, where $\mu(p)$ is the Morse index of p. Moreover, S_p and U_p intersect transversely at only p. We may put orientations of S_p and U_p so that the intersection number $U_p \cap S_p$ is $+1$.

For a generic f, the unstable manifolds of f give a CW-decomposition of M. We denote by $M^k := \bigcup_{\mu(p) \leq k} U_p$ the k-skeleton. Then the connecting homomorphism $\delta_k : H_k(M^k, M^{k-1}; \mathbb{Z}) \to H_{k-1}(M^{k-1}, M^{k-2}; \mathbb{Z})$ satisfy $\delta_{k-1} \circ \delta_k = 0$, and the homology of the chain complex $(H_*(M^*, M^{*-1}; \mathbb{Z}), \delta_*)$ is isomorphic to the singular homology of M. On the other hand, under the natural identification

$$H_k(M^k, M^{k-1}; \mathbb{Z}) \cong \bigoplus_{\mu(p)=k} \mathbb{Z} U_p,$$

the connecting homomorphism can be written as

$$\delta_k U_p = \sum_{\mu(q)=k-1} \sharp(\partial W_p \cap S_q) U_q,$$

where $W_p := \{x \in U_p : f(x) \geq f(p) - \varepsilon\}$, for some small $\varepsilon > 0$ so that W_p is diffeomorphic to a closed ball, and ∂W_p is the boundary of W_p, and $\sharp(\partial W_p \cap S_q)$ is the intersection number of ∂W_p and S_q. Note that this description was essentially given by Milnor in [8], and tells us Morse homology. We define

$$C_k(f) := \bigoplus_{\mu(p)=k} \mathbb{Z}p,$$

which is isomorphic to $H_k(M^k, M^{k-1}; \mathbb{Z})$ by identifying p with U_p. Note that an intersection point $x \in \partial W_p \cap S_q$ corresponds to the unparameterized negative gradient trajectory from p to q passing through x, and we define $\mathcal{M}(p,q)$ to be the set of unparameterized negative gradient trajectories from p to q. Then we define a linear map $\partial_k : C_k(f) \to C_{k-1}(f)$ by

$$\partial_k p := \sum_{\mu(q)=k-1} \sharp\mathcal{M}(p,q)q,$$

which coincides with δ_k by identifying $\partial W_p \cap S_q$ with $\mathcal{M}(p,q)$ as a 0-dimensional oriented compact smooth manifold. Then, we obtain Morse complex $(C_*(f), \partial_*)$, and we call its homology Morse homology, which is isomorphic to the singular homology of M.

The point of closed manifold case is that unstable manifolds give a CW-complex and the boundary operator of Morse complex is nothing but the connecting homomorphism.

Next we review Morse homology of manifolds with boundary, see [1].

Let M be an n-dimensional oriented compact manifold with boundary ∂M. We identify a collar neighborhood of the boundary with $[0,1) \times \partial M$, and denote by r the standard coordinate on the first factor. Take a Riemannian metric g on $M \setminus \partial M$ such that $g|_{(0,1)\times\partial M} = \frac{1}{r} dr \otimes dr + r g_{\partial M}$, where $g_{\partial M}$ is a Riemannian metric on ∂M. Let f be a Morse function on $M \setminus \partial M$ which satisfies the following conditions:

- There is a Morse function $f_{\partial M}$ on ∂M such that $f|_{(0,1)\times\partial M} = r f_{\partial M}$; and
- If γ is a critical point of $f_{\partial M}$, then $f_{\partial M}(\gamma)$ is not equal to zero.

We call $\gamma \in \partial M$ a *positive boundary critical point* if γ is a critical point of $f_{\partial M}$ and $f_{\partial M}(\gamma) > 0$, and similarly, we call $\delta \in \partial M$ a *negative boundary critical point* if δ is a critical point of $f_{\partial M}$ and $f_{\partial M}(\delta) < 0$. On the other hand, we call $p \in M \setminus \partial M$ an *interior critical point* if p is a critical point of f. Note that we always use notation $\gamma, \gamma' \in \partial M$ for positive boundary critical points, $\delta, \delta' \in \partial M$ for negative boundary critical points, and $p, p' \in M \setminus \partial M$ for interior critical points.

On the collar neighborhood $(0,1)\times\partial M$, the gradient vector field X_f with respect to f and g is $r f_{\partial M} \frac{\partial}{\partial r} + X_{f_{\partial M}}$, where $X_{f_{\partial M}}$ is the gradient vector field with respect to $f_{\partial M}$ and $g_{\partial M}$, and we define a vector field $\overline{X}_f$ on M by

$$\overline{X}_f := \begin{cases} X_f, & \text{on } M \setminus \partial M, \\ X_{f_{\partial M}}, & \text{on } \{0\} \times \partial M. \end{cases}$$

We denote by $\overline{\varphi}_t : M \to M$ the isotopy of $-\overline{X}_f$, i.e., $\overline{\varphi}_t$ is given by $d\overline{\varphi}_t/dt = -\overline{X}_f \circ \overline{\varphi}_t$ and $\overline{\varphi}_0(x) = x$.

Remember that, in the closed manifold case, unstable manifolds give a CW-complex. But, in the case of manifolds with boundary, unstable manifolds may *not* give a CW-complex; We would explain this point. Denote by B^k the k-dimensional open ball, by $\overline{B}^k$ the k-dimensional closed ball, and by $\partial\overline{B}^k$ the boundary of $\overline{B}^k$. Moreover, we define $H^k := \{(x_1, \dots, x_k) : x_1^2 + \cdots + x_k^2 < 1, x_k \geq 0\}$ and $\partial H^k := \{(x_1, \dots, x_k) \in H^k : x_k = 0\}$.

As in the closed manifold case, for an interior critical point $p \in M \setminus \partial M$, we define the stable manifold S_p by

$$S_p := \left\{ x \in M : \lim_{t \to +\infty} \overline{\varphi}_t(x) = p \right\} \subset M \setminus \partial M,$$

and similarly, the unstable manifold U_p by

$$U_p := \left\{ x \in M : \lim_{t \to -\infty} \overline{\varphi}_t(x) = p \right\} \subset M \setminus \partial M.$$

Note that S_p is diffeomorphic to $B^{n-\mu(p)}$, and U_p is diffeomorphic to $B^{\mu(p)}$. Moreover, S_p and U_p intersect transversely at only p. We may put orientations of S_p and U_p so that the intersection number $U_p \cap S_p$ is $+1$.

Next, for a positive boundary critical point $\gamma \in \partial M$, we define the stable manifold S_γ by

$$S_\gamma := \left\{ x \in M : \lim_{t \to +\infty} \overline{\varphi}_t(x) = \gamma \right\} \subset M,$$

and the unstable manifold U_γ by

$$U_\gamma := \left\{ x \in M : \lim_{t \to -\infty} \overline{\varphi}_t(x) = \gamma \right\} \subset \partial M.$$

Note that U_γ is diffeomorphic to $B^{\mu(\gamma)}$, where $\mu(\gamma)$ is the Morse index of γ for the Morse function $f_{\partial M} : \partial M \to \mathbb{R}$, and S_γ is diffeomorphic to $H^{n-\mu(\gamma)}$. Moreover, S_γ and U_γ intersect transversely at only $\gamma \in \partial M$. We may put orientations of S_γ and U_γ so that the intersection number $U_\gamma \cap S_\gamma$ is $+1$. Similarly, for a negative boundary critical point $\delta \in \partial M$, we define the stable manifold S_δ by

$$S_\delta := \left\{ x \in M : \lim_{t \to +\infty} \overline{\varphi}_t(x) = \delta \right\} \subset \partial M,$$

and the unstable manifold U_δ by

$$U_\delta := \left\{ x \in M : \lim_{t \to -\infty} \overline{\varphi}_t(x) = \delta \right\} \subset M.$$

Note that S_δ is diffeomorphic to $B^{n-1-\mu(\delta)}$, where $\mu(\delta)$ is the Morse index of δ for the Morse function $f_{\partial M} : \partial M \to \mathbb{R}$, and U_δ is diffeomorphic to $H^{\mu(\delta)+1}$. Moreover, S_δ and U_δ intersect transversely at only $\delta \in \partial M$. We may put orientations of S_δ and U_δ so that the intersection number $U_\delta \cap S_\delta$ is $+1$.

Note that, since U_δ is not diffeomorphic to an open ball, the unstable manifolds do not give a CW-decomposition of M if $f_{\partial M}$ has negative boundary critical points. Moreover, U_γ may be attached to the same dimensional U_δ, which is another reason why the unstable manifolds do not give a CW-decomposition of M. But we have some stratification of M, and obtain a chain complex whose homology is isomorphic to $H_*(M;\mathbb{Z})$, the absolute singular homology of M. We would explain this chain complex next.

Let f be generic. For a positive boundary critical point $\gamma \in \partial M$, we fix a diffeomorphism $i_\gamma : B^{\mu(\gamma)} \to U_\gamma \subset \partial M$, and extend i_γ to be a continuous map $\overline{i}_\gamma : \overline{B}^{\mu(\gamma)} \to \partial M$. Note that $\overline{i}_\gamma$ may not be injective on $\partial \overline{B}^{\mu(\gamma)}$. Let $\delta_1, \dots, \delta_N \in \partial M$ be the negative boundary critical points with $\mu(\delta_1) = \cdots = \mu(\delta_N) = \mu(\gamma) - 1$. We also fix diffeomorphisms $i_{\delta_j} : H^{\mu(\delta_j)+1} \to U_{\delta_j} \subset M$, for $j = 1, \dots, N$. Suppose that there are k_j negative gradient trajectories from γ to δ_j. (k_j might be

0.) Let $H_1^{\mu(\delta_j)+1},\dots,H_{k_j}^{\mu(\delta_j)+1}$ be k_j-copies of $H^{\mu(\delta_j)+1}$, for $j=1,\dots,N$. We write $\bar{i}_\gamma^{-1}(U_{\delta_j}\cap\partial M)=A_{j1}\sqcup\cdots\sqcup A_{jk_j}\subset\partial\overline{B}^{\mu(\gamma)}$, where A_{ji} is a connected component. Then we identify $x\in A_{ji}$ with $y\in\partial H_i^{\mu(\delta_j)+1}$ if $\bar{i}_\gamma(x)=i_{\delta_j}(y)$, and we attach $H_1^{\mu(\delta_1)+1},\dots,H_{k_N}^{\mu(\delta_N)+1}$ to $\overline{B}^{\mu(\gamma)}$ by this identification. Define e_γ to be the interior of $\overline{B}^{\mu(\gamma)}\cup H_1^{\mu(\delta_1)+1}\cup\cdots\cup H_{k_N}^{\mu(\delta_N)+1}$, which is homeomorphic to the $\mu(\gamma)$-dimensional open ball, and define a continuous map $I_\gamma:e_\gamma\to M$ whose restriction on $B^{\mu(\gamma)},H_1^{\mu(\delta_1)+1},\dots,H_{k_N}^{\mu(\delta_N)+1}$ is $i_\gamma,i_{\delta_1},\dots,i_{\delta_N}$, respectively. Note that I_γ is not injective on $H_1^{\mu(\delta_j)+1}\cup\cdots\cup H_{k_j}^{\mu(\delta_j)+1}$ if $k_j\geq 2$. Then we define

$$M^k:=\bigcup_{\mu(p)\leq k}U_p\cup\bigcup_{\mu(\gamma)\leq k}U_\gamma\cup\bigcup_{\mu(\delta)\leq k-1}U_\delta,$$

where p is an interior critical point, γ is a positive boundary critical point, and δ is a negative boundary critical point. Note that M^k is homotopic to

$$\bigcup_{\mu(p)\leq k}U_p\cup\bigcup_{\mu(\gamma)\leq k}I_\gamma(e_\gamma).$$

The connecting homomorphism $\delta_k: H_k(M^k,M^{k-1};\mathbb{Z})\to H_{k-1}(M^{k-1},M^{k-2};\mathbb{Z})$ satisfy $\delta_{k-1}\circ\delta_k=0$, and the homology of $(H_*(M^*,M^{*-1};\mathbb{Z}),\delta_*)$ is isomorphic to $H_*(M;\mathbb{Z})$, see [1]. On the other hand, under the natural identification

$$H_k(M^k,M^{k-1};\mathbb{Z})\cong\bigoplus_{\mu(p)=k}\mathbb{Z}U_p\oplus\bigoplus_{\mu(\gamma)=k}\mathbb{Z}I_\gamma,$$

the connecting homomorphism can be written as

$$\begin{aligned}\delta_kU_p&=\sum_{\mu(p')=k-1}\sharp(\partial W_p\cap S_{p'})U_{p'}+\sum_{\mu(\gamma')=k-1}\sharp(\partial W_p\cap S_{\gamma'})I_{\gamma'},\\ \delta_kI_\gamma&=\sum_{\mu(\gamma')=k-1}\sharp(\partial W_\gamma\cap S_{\gamma'})I_{\gamma'}+\sum_{\substack{\mu(\delta)=k-1\\ \mu(p')=k-1}}\sharp(\partial W_\gamma\cap S_\delta)\sharp(\partial W_\delta\cap S_{p'})U_{p'},\end{aligned}$$

where p,p' are interior critical points, γ,γ' are positive boundary critical points, and δ is a negative boundary critical point. Then, this description tells us our Morse homology. We define

$$C_k(f):=\bigoplus_{\mu(p)=k}\mathbb{Z}p\oplus\bigoplus_{\mu(\gamma)=k}\mathbb{Z}\gamma,$$

which is isomorphic to $H_k(M^k,M^{k-1};\mathbb{Z})$ by identifying p and γ with U_p and I_γ, respectively. We define $\mathcal{M}(p,p'),\mathcal{M}(p,\gamma'),\mathcal{M}_N(\gamma,\gamma'),\mathcal{M}_N(\gamma,\delta),\mathcal{M}(\delta,p')$ to be the sets of unparameterized negative gradient trajectories from p to p' in M, p to γ' in M, γ to γ' in ∂M, γ to δ in ∂M, δ to p' in M, respectively. Then we define a linear map $\partial_k:C_k(f)\to C_{k-1}(f)$ by

$$\begin{aligned}\partial_kp&:=\sum_{\mu(p')=k-1}\sharp\mathcal{M}(p,p')p'+\sum_{\mu(\gamma')=k-1}\sharp\mathcal{M}(p,\gamma')\gamma',\\ \partial_k\gamma&:=\sum_{\mu(\gamma')=k-1}\sharp\mathcal{M}_N(\gamma,\gamma')\gamma'+\sum_{\substack{\mu(\delta)=k-1\\ \mu(p')=k-1}}\sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,p')p',\end{aligned}$$

which coincides with δ_k by identifying $\partial W_* \cap S_{*'}$ with $\mathcal{M}(*, *')$ as a 0-dimensional oriented compact smooth manifold. This is our Morse complex, and its homology is isomorphic to $H_*(M;\mathbb{Z})$, the absolute singular homology, see [1].

Note that we may also prove $\partial_{k-1} \circ \partial_k = 0$ by observing the boundary of 1-dimensional moduli spaces of unparameterized negative gradient trajectories, see Section 4 and [1], which is very important for Floer homology of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end, see Section 5 and [1].

There are some remarks about other related works; In [6] Kronheimer–Mrowka also studied Morse homology of manifolds with boundary. They considered the double of a manifold with boundary and involution invariant Morse functions. Then, they obtained similar Morse complex, and applied their Morse homology to Seiberg–Witten Floer theory. In [7] F. Laudenbach also studied Morse homology of manifolds with boundary. He considered pseudo-gradient vector fields and their trajectories, and then obtained similar Morse homology.

3. Cup products

In this section, we observe cup products on Morse homology of manifolds with boundary. But, before manifolds with boundary, we briefly review cup products on Morse homology of closed manifolds, see [5].

In the previous section, we saw that a single Morse function tells us the singular homology. On the other hand, in [5] Fukaya found that we need three Morse functions to describe cup products in terms of Morse homology.

Let M be an n-dimensional oriented closed manifold, and g a Riemannian metric on M. Let f_i be a Morse function on M, for $i = 1, 2, 3$. For a critical point p of f_i, we denote by $S_p^{f_i}$ and $U_p^{f_i}$ the stable manifold and the unstable manifold, respectively. Let $M_i^k := \bigcup_{\mu(p)\le k} U_p^{f_i}$ be the k-skeleton with respect to f_i.

Suppose that f_1, f_2, f_3 are generic so that $U_{p_1}^{f_1}, U_{p_2}^{f_2}$ and $S_{p_3}^{f_3}$ intersect transversely. Then, if $\mu(p_1) + \mu(p_2) - n = \mu(p_3)$, $U_{p_1}^{f_1} \cap U_{p_2}^{f_2} \cap S_{p_3}^{f_3}$ is a 0-dimensional oriented compact smooth manifold, and we define a linear map

$$\begin{aligned} m_2 : H_{k_1}(M_1^{k_1}, M_1^{k_1-1};\mathbb{Z}) \otimes H_{k_2}(M_2^{k_2}, M_2^{k_2-1};\mathbb{Z}) \\ \to H_{k_1+k_2-n}(M_3^{k_1+k_2-n}, M_3^{k_1+k_2-n-1};\mathbb{Z}) \end{aligned}$$

by

$$m_2(U_{p_1}^{f_1} \otimes U_{p_2}^{f_2}) := \sum_{\mu(p_3)=k_1+k_2-n} \sharp(U_{p_1}^{f_1} \cap U_{p_2}^{f_2} \cap S_{p_3}^{f_3}) U_{p_3}^{f_3},$$

where $\sharp(U_{p_1}^{f_1} \cap U_{p_2}^{f_2} \cap S_{p_3}^{f_3})$ is the number of the points in $U_{p_1}^{f_1} \cap U_{p_2}^{f_2} \cap S_{p_3}^{f_3}$ with sign, where the sign comes from the intersection number. Then Fukaya essentially proved the following theorem, see [5]:

Theorem 3.1 (Fukaya [5]). (1) *We denote by $\delta^{f_1}, \delta^{f_2}$ and δ^{f_3} the connecting homomorphisms for f_1, f_2 and f_3, respectively. Then we obtain the Leibniz rule: (We omit the sign convention.)*

$$\delta^{f_3} m_2(U_{p_1}^{f_1} \otimes U_{p_2}^{f_2}) = m_2(\delta^{f_1} U_{p_1}^{f_1} \otimes U_{p_2}^{f_2}) \pm m_2(U_{p_1}^{f_1} \otimes \delta^{f_2} U_{p_2}^{f_2}).$$

(2) *This m_2 gives the cup product.*

Next we *heuristically* describe this m_2 in terms of gradient trees. Note that an intersection point $x \in U^{f_1}_{p_1} \cap U^{f_2}_{p_2} \cap S^{f_3}_{p_3}$ corresponds to the gradient tree (l_1, l_2, l_3) such that

- $l_1 : (-\infty, 0] \to M$ satisfies $dl_1/dt = -X_{f_1} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = p_1$ and $l_1(0) = x$;
- $l_2 : (-\infty, 0] \to M$ satisfies $dl_2/dt = -X_{f_2} \circ l_2$, and $\lim_{t\to-\infty} l_2(t) = p_2$ and $l_2(0) = x$; and
- $l_3 : [0, \infty) \to M$ satisfies $dl_3/dt = -X_{f_3} \circ l_3$, and $l_3(0) = x$ and $\lim_{t\to\infty} l_3(t) = p_3$.

We denote by $\mathcal{M}(p_1, p_2, p_3)$ the set of such gradient trees (l_1, l_2, l_3). Then, under the identification $H_{k_i}(M^{k_i}_i, M^{k_i-1}_i; \mathbb{Z}) \cong C_{k_i}(f_i)$, we may redefine the linear map $m_2 : C_{k_1}(f_1) \otimes C_{k_2}(f_2) \to C_{k_1+k_2-n}(f_3)$ by

$$m_2(p_1 \otimes p_2) := \sum_{\mu(p_3)=\mu(p_1)+\mu(p_2)-n} \sharp\mathcal{M}(p_1, p_2, p_3)p_3.$$

Note that we may also prove the Leibniz rules by observing the boundary of 1-dimensional moduli spaces of gradient trees, which is very important for Fukaya category, see Section 4 and [5].

Next we observe cup products on Morse homology of manifolds with boundary. In the case of closed manifolds, we used unstable manifolds to obtain the cup products. But, in the case of manifolds with boundary, we use the unstable manifolds of interior critical points and $I_\gamma : e_\gamma \to M$ of positive boundary critical points.

Let M be an n-dimensional oriented compact manifold with boundary ∂M. We fix a collar neighborhood and a Riemannian metric on $M \setminus \partial M$ as in Section 2. Let $f_i : M \setminus \partial M \to \mathbb{R}$ be a Morse function which satisfies the same conditions as in Section 2, for $i = 1, 2, 3$. We denote by $f_{i\partial M}$ the boundary Morse function of f_i, for $i = 1, 2, 3$. For an interior critical point $p \in M \setminus \partial M$ of f_i, we denote by $S^{f_i}_p \subset M \setminus \partial M$ and $U^{f_i}_p \subset M \setminus \partial M$ the stable manifold and the unstable manifold, respectively. For a positive boundary critical point $\gamma \in \partial M$ of $f_{i\partial M}$, we denote by $S^{f_i}_\gamma \subset M$ and $U^{f_i}_\gamma \subset \partial M$ the stable manifold and the unstable manifold, respectively, and similarly, for a negative boundary critical point $\delta \in \partial M$ of $f_{i\partial M}$, we denote by $S^{f_i}_\delta \subset \partial M$ and $U^{f_i}_\delta \subset M$ the stable manifold and the unstable manifold, respectively. Remember that, in the previous section, we introduce the notation $I_\gamma : e_\gamma \to M$ for a positive boundary critical point γ. Then, we use the notation $I^i_\gamma : e^i_\gamma \to M$ for a positive boundary critical point γ of $f_{i\partial M}$, for $i = 1, 2, 3$.

Note that $U^{f_1}_{\gamma_1} \cap U^{f_2}_{\gamma_2} \subset \partial M$, and if we push $U^{f_2}_{\gamma_2}$ into $M \setminus \partial M$ slightly, then the intersection points of $U^{f_1}_{\gamma_1}$ and the pushed $U^{f_2}_{\gamma_2}$ disappear, which means that the intersection of $U^{f_1}_{\gamma_1}$ and $U^{f_2}_{\gamma_2}$ is not transversal. So we need some trick to get correct intersection numbers as follows.

Let $\lambda_\varepsilon : [0, 1] \to [\varepsilon, 1]$ be a diffeomorphism so that $\lambda(0) = \varepsilon$ and the restriction of λ_ε on $[2\varepsilon, 1]$ is the identity, for small $\varepsilon > 0$. Then, we define a smooth map $\psi_\varepsilon : M \to M$ by

$$\psi_\varepsilon(x) := \begin{cases} x, & \text{for } x \in M \setminus [0,1] \times \partial M, \\ (\lambda_\varepsilon(r), y), & \text{for } x = (r, y) \in [0,1] \times \partial M. \end{cases}$$

Suppose that f_1, f_2, f_3 and a small $\varepsilon > 0$ are generic so that

- $U^{f_1}_{p_1}, \psi_\varepsilon(U^{f_2}_{p_2})$ and $S^{f_3}_{p_3}$ intersect transversely;
- $U^{f_1}_{p_1}, \psi_\varepsilon(U^{f_2}_{p_2})$ and $S^{f_3}_{\gamma_3}$ intersect transversely;

- $U_{p_1}^{f_1}, \psi_\varepsilon \circ I_{\gamma_2}^2 : e_{\gamma_2}^2 \to M$ and $S_{p_3}^{f_3}$ intersect transversely;
- $U_{p_1}^{f_1}, \psi_\varepsilon \circ I_{\gamma_2}^2 : e_{\gamma_2}^2 \to M$ and $S_{\gamma_3}^{f_3}$ intersect transversely;
- $I_{\gamma_1}^1 : e_{\gamma_1}^1 \to M, \psi_\varepsilon(U_{p_2}^{f_2})$ and $S_{p_3}^{f_3}$ intersect transversely;
- $I_{\gamma_1}^1 : e_{\gamma_1}^1 \to M, \psi_\varepsilon(U_{p_2}^{f_2})$ and $S_{\gamma_3}^{f_3}$ intersect transversely;
- $I_{\gamma_1}^1 : e_{\gamma_1}^1 \to M, \psi_\varepsilon \circ I_{\gamma_2}^2 : e_{\gamma_2}^2 \to M$ and $S_{p_3}^{f_3}$ intersect transversely; and
- $I_{\gamma_1}^1 : e_{\gamma_1}^1 \to M, \psi_\varepsilon \circ I_{\gamma_2}^2 : e_{\gamma_2}^2 \to M$ and $S_{\gamma_3}^{f_3}$ intersect transversely.

In fact we may take such generic f_1, f_2, f_3 and a small ε by the standard transversality argument in Morse theory. Then the following fiber products are 0-dimensional oriented compact smooth manifolds, where the orientations come from the intersection numbers. Note that we use notation $i_* : B^{\mu(*)} \to U_*^*$ and $j_* : B^{n-\mu(*)} \to S_*^*$ for diffeomorphisms:

- $\{(x_1, x_2, x_3) \in B^{\mu(p_1)} \times B^{\mu(p_2)} \times B^{n-\mu(p_3)} : i_{p_1}(x_1) = \psi_\varepsilon \circ i_{p_2}(x_2) = j_{p_3}(x_3)\}$, for p_1, p_2 and p_3 with $\mu(p_1) + \mu(p_2) - n = \mu(p_3)$;
- $\{(x_1, x_2, x_3) \in B^{\mu(p_1)} \times B^{\mu(p_2)} \times B^{n-\mu(\gamma_3)} : i_{p_1}(x_1) = \psi_\varepsilon \circ i_{p_2}(x_2) = j_{\gamma_3}(x_3)\}$, for p_1, p_2 and γ_3 with $\mu(p_1) + \mu(p_2) - n = \mu(\gamma_3)$;
- $\{(x_1, x_2, x_3) \in B^{\mu(p_1)} \times e_{\gamma_2}^2 \times B^{n-\mu(p_3)} : i_{p_1}(x_1) = \psi_\varepsilon \circ I_{\gamma_2}^2(x_2) = j_{p_3}(x_3)\}$, for p_1, γ_2 and p_3 with $\mu(p_1) + \mu(\gamma_2) - n = \mu(p_3)$;
- $\{(x_1, x_2, x_3) \in B^{\mu(p_1)} \times e_{\gamma_2}^2 \times B^{n-\mu(\gamma_3)} : i_{p_1}(x_1) = \psi_\varepsilon \circ I_{\gamma_2}^2(x_2) = j_{\gamma_3}(x_3)\}$, for p_1, γ_2 and γ_3 with $\mu(p_1) + \mu(\gamma_2) - n = \mu(\gamma_3)$;
- $\{(x_1, x_2, x_3) \in e_{\gamma_1}^1 \times B^{\mu(p_2)} \times B^{n-\mu(p_3)} : I_{\gamma_1}^1(x_1) = \psi_\varepsilon \circ i_{p_2}(x_2) = j_{p_3}(x_3)\}$, for γ_1, p_2 and p_3 with $\mu(\gamma_1) + \mu(p_2) - n = \mu(p_3)$;
- $\{(x_1, x_2, x_3) \in e_{\gamma_1}^1 \times B^{\mu(p_2)} \times B^{n-\mu(\gamma_3)} : I_{\gamma_1}^1(x_1) = \psi_\varepsilon \circ i_{p_2}(x_2) = j_{\gamma_3}(x_3)\}$, for γ_1, p_2 and γ_3 with $\mu(\gamma_1) + \mu(p_2) - n = \mu(\gamma_3)$;
- $\{(x_1, x_2, x_3) \in e_{\gamma_1}^1 \times e_{\gamma_2}^2 \times B^{n-\mu(p_3)} : I_{\gamma_1}^1(x_1) = \psi_\varepsilon \circ I_{\gamma_2}^2(x_2) = j_{p_3}(x_3)\}$, for γ_1, γ_2 and p_3 with $\mu(\gamma_1) + \mu(\gamma_2) - n = \mu(p_3)$; and
- $\{(x_1, x_2, x_3) \in e_{\gamma_1}^1 \times e_{\gamma_2}^2 \times B^{n-\mu(\gamma_3)} : I_{\gamma_1}^1(x_1) = \psi_\varepsilon \circ I_{\gamma_2}^2(x_2) = j_{\gamma_3}(x_3)\}$, for γ_1, γ_2 and γ_3 with $\mu(\gamma_1) + \mu(\gamma_2) - n = \mu(\gamma_3)$.

We denote by $n(p_1, p_2, p_3), n(p_1, p_2, \gamma_3), \dots$ the number of the points of the fiber products above with sign, where the sign comes from the intersection number.

Under the identification $H_{k_i}(M_i^{k_i}, M_i^{k_i-1}; \mathbb{Z}) \cong \bigoplus_{\mu(p_i)=k_i} \mathbb{Z}U_{p_i}^{f_i} \oplus \bigoplus_{\mu(\gamma)=k_i} \mathbb{Z}I_{\gamma_i}^i$, for $i = 1, 2, 3$, we define a linear map

$$\begin{aligned} m_2 : H_{k_1}(M_1^{k_1}, M_1^{k_1-1}; \mathbb{Z}) &\otimes H_{k_2}(M_2^{k_2}, M_2^{k_2-1}; \mathbb{Z}) \\ &\to H_{k_1+k_2-n}(M_3^{k_1+k_2-n}, M_3^{k_1+k_2-n-1}; \mathbb{Z}) \end{aligned}$$

by

$$\begin{aligned} m_2(U_{p_1}^{f_1} \otimes U_{p_2}^{f_2}) &:= \sum_{\mu(p_3)=k_1+k_2-n} n(p_1, p_2, p_3) U_{p_3}^{f_3} + \sum_{\mu(\gamma_3)=k_1+k_2-n} n(p_1, p_2, \gamma_3) I_{\gamma_3}^3, \\ m_2(U_{p_1}^{f_1} \otimes I_{\gamma_2}^2) &:= \sum_{\mu(p_3)=k_1+k_2-n} n(p_1, \gamma_2, p_3) U_{p_3}^{f_3} + \sum_{\mu(\gamma_3)=k_1+k_2-n} n(p_1, \gamma_2, \gamma_3) I_{\gamma_3}^3, \\ m_2(I_{\gamma_1}^1 \otimes U_{p_2}^{f_2}) &:= \sum_{\mu(p_3)=k_1+k_2-n} n(\gamma_1, p_2, p_3) U_{p_3}^{f_3} + \sum_{\mu(\gamma_3)=k_1+k_2-n} n(\gamma_1, p_2, \gamma_3) I_{\gamma_3}^3, \\ m_2(I_{\gamma_1}^1 \otimes I_{\gamma_2}^2) &:= \sum_{\mu(p_3)=k_1+k_2-n} n(\gamma_1, \gamma_2, p_3) U_{p_3}^{f_3} + \sum_{\mu(\gamma_3)=k_1+k_2-n} n(\gamma_1, \gamma_2, \gamma_3) I_{\gamma_3}^3, \end{aligned}$$

where p_1, p_2 and p_3 are interior critical points of f_1, f_2 and f_3, respectively, and γ_1, γ_2 and γ_3 are positive boundary critical points of f_1, f_2 and f_3, respectively. Then we obtain the following theorem:

Theorem 3.2. (1) *We denote by $\delta^{f_1}, \delta^{f_2}$ and δ^{f_3} the connecting homomorphisms for f_1, f_2 and f_3, respectively. Then we obtain the Leibniz rules: (We omit the sign convention.)*

$$\begin{aligned}
\delta^{f_3} m_2(U^{f_1}_{p_1} \otimes U^{f_2}_{p_2}) &= m_2(\delta^{f_1} U^{f_1}_{p_1} \otimes U^{f_2}_{p_2}) \pm m_2(U^{f_1}_{p_1} \otimes \delta^{f_2} U^{f_2}_{p_2}),\\
\delta^{f_3} m_2(U^{f_1}_{p_1} \otimes I^2_{\gamma_2}) &= m_2(\delta^{f_1} U^{f_1}_{p_1} \otimes I^2_{\gamma_2}) \pm m_2(U^{f_1}_{p_1} \otimes \delta^{f_2} I^2_{\gamma_2}),\\
\delta^{f_3} m_2(I^1_{\gamma_1} \otimes U^{f_2}_{p_2}) &= m_2(\delta^{f_1} I^1_{\gamma_1} \otimes U^{f_2}_{p_2}) \pm m_2(I^1_{\gamma_1} \otimes \delta^{f_2} U^{f_2}_{p_2}),\\
\delta^{f_3} m_2(I^1_{\gamma_1} \otimes I^2_{\gamma_2}) &= m_2(\delta^{f_1} I^1_{\gamma_1} \otimes I^2_{\gamma_2}) \pm m_2(I^1_{\gamma_1} \otimes \delta^{f_2} I^2_{\gamma_2}).
\end{aligned}$$

(2) *This m_2 gives the cup product.*

Proof. We may think $i_p : B^{\mu(p)} \to M, \psi_\varepsilon \circ i_p : B^{\mu(p)} \to M, I_\gamma : e_\gamma \to M$ and $\psi_\varepsilon \circ I_\gamma : e_\gamma \to M$ as chains, and hence m_2 satisfies the Leibniz rules as in the case of closed manifolds. ☐

Next we *heuristically* describe our m_2 in terms of gradient trees. We have to fix all $n(*_1, *_2, *_3)$, 8 types! Note that we always use notation, for $i = 1, 2, 3$,

- $p_i, p'_i, p''_i \in M \setminus \partial M$ for interior critical points of f_i;
- $\gamma_i, \gamma'_i, \gamma''_i \in \partial M$ for positive boundary critical points of $f_{i\partial M}$; and
- $\delta_i, \delta'_i, \delta''_i \in \partial M$ for negative boundary critical points of $f_{i\partial M}$.

First, we fix $n(p_1, p_2, p_3)$. Let p_1, p_2 and p_3 be interior critical points of f_1, f_2 and f_3, respectively. Suppose $\mu(p_1) + \mu(p_2) - n = \mu(p_3)$. For small $\varepsilon > 0$, we define

$$I_\varepsilon(p_1, p_2, p_3) :=$$
$$\left\{(x_1, x_2, x_3) \in B^{\mu(p_1)} \times B^{\mu(p_2)} \times B^{n-\mu(p_3)} : i_{p_1}(x_1) = \psi_\varepsilon \circ i_{p_2}(x_2) = j_{p_3}(x_3)\right\}.$$

Since $\varepsilon > 0$ is small enough, for each $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon}) \in I_\varepsilon(p_1, p_2, p_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s := (x_{1s}, x_{2s}, x_{3s}) \in I_s(p_1, p_2, p_3)$. Note that (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree (l_{1s}, l_{2s}, l_{3s}) such that

- $l_{1s} : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = p_1$ and $l_{1s}(0) = i_{p_1}(x_{1s})$;
- $l_{2s} : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_{2s}/dt = -X_{f_2} \circ l_{2s}$, and $\lim_{t\to-\infty} l_{2s}(t) = p_2$ and $l_{2s}(0) = i_{p_2}(x_{2s})$; and
- $l_{3s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_3} \circ l_{3s}$, and $l_{3s}(0) = j_{p_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = p_3$.

We define $(w_1, w_2, w_3) := \lim_{s\to 0}(x_{1s}, x_{2s}, x_{3s})$. Note that $i_{p_1}(w_1) = i_{p_2}(w_2) = j_{p_3}(w_3)$. Then, when $s \to 0$, (l_{1s}, l_{2s}, l_{3s}) converges to (l_1, l_2, l_3) such that

- $l_1 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_1/dt = -X_{f_1} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = p_1$ and $l_1(0) = i_{p_1}(w_1)$;
- $l_2 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_2/dt = -X_{f_2} \circ l_2$, and $\lim_{t\to-\infty} l_2(t) = p_2$ and $l_2(0) = i_{p_2}(w_2)$; and
- $l_3 : [0, \infty) \to M \setminus \partial M$ satisfies $dl_3/dt = -X_{f_3} \circ l_3$, and $l_3(0) = j_{p_3}(w_3)$ and $\lim_{t\to\infty} l_3(t) = p_3$.

We denote by $\mathcal{M}(p_1,p_2,p_3)$ the set of such gradient trees (l_1,l_2,l_3). Then we obtain

$$n(p_1,p_2,p_3)=\sharp\mathcal{M}(p_1,p_2,p_3).$$

Secondly, we fix $n(p_1,p_2,\gamma_3)$. Let p_1,p_2 be interior critical points of f_1,f_2, respectively, and γ_3 a positive boundary critical point of $f_{3\partial M}$. Suppose $\mu(p_1)+\mu(p_2)-n=\mu(\gamma_3)$. For small $\varepsilon>0$, we define

$$I_\varepsilon(p_1,p_2,\gamma_3):=\left\{(x_1,x_2,x_3)\in B^{\mu(p_1)}\times B^{\mu(p_2)}\times B^{n-\mu(\gamma_3)}:i_{p_1}(x_1)=\psi_\varepsilon\circ i_{p_2}(x_2)=j_{\gamma_3}(x_3)\right\}.$$

Since $\varepsilon>0$ is small enough, for each $(x_{1\varepsilon},x_{2\varepsilon},x_{3\varepsilon})\in I_\varepsilon(p_1,p_2,\gamma_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s:=(x_{1s},x_{2s},x_{3s})\in I_s(p_1,p_2,\gamma_3)$. Note that (x_{1s},x_{2s},x_{3s}) corresponds to a gradient tree (l_{1s},l_{2s},l_{3s}) such that

- $l_{1s}:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_{1s}/dt=-X_{f_1}\circ l_{1s}$, and $\lim_{t\to-\infty}l_{1s}(t)=p_1$ and $l_{1s}(0)=i_{p_1}(x_{1s})$;
- $l_{2s}:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_{2s}/dt=-X_{f_2}\circ l_{2s}$, and $\lim_{t\to-\infty}l_{2s}(t)=p_2$ and $l_{2s}(0)=i_{p_2}(x_{2s})$; and
- $l_{3s}:[0,\infty)\to M\setminus\partial M$ satisfies $dl_{3s}/dt=-X_{f_3}\circ l_{3s}$, and $l_{3s}(0)=j_{\gamma_3}(x_{3s})$ and $\lim_{t\to\infty}l_{3s}(t)=\gamma_3$.

We define $(w_1,w_2,w_3):=\lim_{s\to0}(x_{1s},x_{2s},x_{3s})$. Note that $i_{p_1}(w_1)=i_{p_2}(w_2)=j_{\gamma_3}(w_3)$. Then, when $s\to0$, (l_{1s},l_{2s},l_{3s}) converges to (l_1,l_2,l_3) such that

- $l_1:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_1/dt=-X_{f_1}\circ l_1$, and $\lim_{t\to-\infty}l_1(t)=p_1$ and $l_1(0)=i_{p_1}(w_1)$;
- $l_2:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_2/dt=-X_{f_2}\circ l_2$, and $\lim_{t\to-\infty}l_2(t)=p_2$ and $l_2(0)=i_{p_2}(w_2)$; and
- $l_3:[0,\infty)\to M\setminus\partial M$ satisfies $dl_3/dt=-X_{f_3}\circ l_3$, and $l_3(0)=j_{\gamma_3}(w_3)$ and $\lim_{t\to\infty}l_3(t)=\gamma_3$.

We denote by $\mathcal{M}(p_1,p_2,\gamma_3)$ the set of such gradient trees (l_1,l_2,l_3). Then we obtain

$$n(p_1,p_2,\gamma_3)=\sharp\mathcal{M}(p_1,p_2,\gamma_3).$$

Thirdly, we fix $n(p_1,\gamma_2,p_3)$. Let p_1,p_3 be interior critical points of f_1,f_3, respectively, and γ_2 a positive boundary critical point of $f_{2\partial M}$. Suppose $\mu(p_1)+\mu(\gamma_2)-n=\mu(p_3)$. For small $\varepsilon>0$, we define

$$I_\varepsilon(p_1,\gamma_2,p_3):=\left\{(x_1,x_2,x_3)\in B^{\mu(p_1)}\times e^2_{\gamma_2}\times B^{n-\mu(p_3)}:i_{p_1}(x_1)=\psi_\varepsilon\circ I^2_{\gamma_2}(x_2)=j_{p_3}(x_3)\right\}.$$

Since $\varepsilon>0$ is small enough, for each $(x_{1\varepsilon},x_{2\varepsilon},x_{3\varepsilon})\in I_\varepsilon(p_1,\gamma_2,p_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s:=(x_{1s},x_{2s},x_{3s})\in I_s(p_1,\gamma_2,p_3)$. There are two possibilities: First $x_{2s}\in B^{\mu(\gamma_2)}\subset e^2_{\gamma_2}$, and secondly $x_{2s}\in H_1^{\mu(\delta_1)+1}\cup\cdots\cup H_{k_N}^{\mu(\delta_N)+1}\subset e^2_{\gamma_2}$. Suppose $x_{2s}\in B^{\mu(\gamma_2)}$, and then (x_{1s},x_{2s},x_{3s}) corresponds to a gradient tree (l_{1s},l_{2s},l_{3s}) such that

- $l_{1s}:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_{1s}/dt=-X_{f_1}\circ l_{1s}$, and $\lim_{t\to-\infty}l_{1s}(t)=p_1$ and $l_{1s}(0)=i_{p_1}(x_{1s})$;

- $l_{2s} : (-\infty, 0] \to \partial M$ satisfies $dl_{2s}/dt = -X_{f_{2\partial M}} \circ l_{2s}$, and $\lim_{t\to-\infty} l_{2s}(t) = \gamma_2$ and $l_{2s}(0) = I^2_{\gamma_2}(x_{2s})$; and
- $l_{3s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_3} \circ l_{3s}$, and $l_{3s}(0) = j_{p_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = p_3$.

We define $(w_1, w_2, w_3) := \lim_{s\to 0}(x_{1s}, x_{2s}, x_{3s})$. Then, when $s \to 0$, l_{1s} converges to a broken trajectory (l_0, l_1) such that

- l_0 is a negative gradient trajectory of f_1 from p_1 to γ_1', where γ_1' is a positive boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1') + 1 = \mu(p_1)$; and
- $l_1 : (-\infty, 0] \to \partial M$ satisfies $dl_1/dt = -X_{f_{1\partial M}} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = \gamma_1'$ and $l_1(0) = I^2_{\gamma_2}(w_2)$,

l_{2s} converges to l_2 such that

- $l_2 : (-\infty, 0] \to \partial M$ satisfies $dl_2/dt = -X_{f_{2\partial M}} \circ l_2$, and $\lim_{t\to-\infty} l_2(t) = \gamma_2$ and $l_2(0) = I^2_{\gamma_2}(w_2)$,

and l_{3s} converges to a broken trajectory (l_3, l_4) such that

- $l_3 : [0, \infty) \to \partial M$ satisfies $dl_3/dt = -X_{f_{3\partial M}} \circ l_3$, and $l_3(0) = I^2_{\gamma_2}(w_2)$ and $\lim_{t\to\infty} l_3(t) = \delta_3'$, where δ_3' is a negative boundary critical point of $f_{3\partial M}$ with $\mu(\delta_3') = \mu(p_3)$; and
- l_4 is a negative gradient trajectory of f_3 from δ_3' to p_3.

We denote by $\mathcal{M}(p_1, \gamma_1')$ the set of such unparameterized negative gradient trajectories l_0, by $\mathcal{M}_N(\gamma_1', \gamma_2, \delta_3')$ the set of such gradient trees (l_1, l_2, l_3), and by $\mathcal{M}(\delta_3', p_3)$ the set of such unparameterized negative gradient trajectories l_4. Then, if $x_{2\varepsilon} \in B^{\mu(\gamma_2)} \subset e^2_{\gamma_2}$, we may identify the set of such $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon})$ with

$$\bigcup_{\gamma_1', \delta_3'} \mathcal{M}(p_1, \gamma_1') \times \mathcal{M}_N(\gamma_1', \gamma_2, \delta_3') \times \mathcal{M}(\delta_3', p_3).$$

Next, suppose $x_{2s} \in H_1^{\mu(\delta_1)+1} \cup \cdots \cup H_{k_N}^{\mu(\delta_N)+1}$, and then (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree $(l_{1s}, l_{2s}, l_{3s}, l_{4s})$ such that

- $l_{1s} : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = p_1$ and $l_{1s}(0) = i_{p_1}(x_{1s})$;
- l_{2s} is a negative gradient trajectory of $f_{2\partial M}$ from γ_2 to δ_2', where δ_2' is a negative boundary critical point of $f_{2\partial M}$ with $\mu(\gamma_2) = \mu(\delta_2') + 1$;
- $l_{3s} : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_2} \circ l_{3s}$, and $\lim_{t\to-\infty} l_{3s}(t) = \delta_2'$ and $l_{3s}(0) = I^2_{\gamma_2}(x_{2s})$; and
- $l_{4s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{4s}/dt = -X_{f_3} \circ l_{4s}$, and $l_{4s}(0) = j_{p_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = p_3$.

We define $(w_1, w_2, w_3) := \lim_{s\to 0}(x_{1s}, x_{2s}, x_{3s})$. Then, when $s \to 0$, $(l_{1s}, l_{2s}, l_{3s}, l_{4s})$ converges to (l_1, l_2, l_3, l_4) such that

- $l_1 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_1/dt = -X_{f_1} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = p_1$ and $l_1(0) = i_{p_1}(w_1)$;
- l_2 is a negative gradient trajectory of $f_{2\partial M}$ from γ_2 to δ_2';
- $l_3 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_3/dt = -X_{f_2} \circ l_3$, and $\lim_{t\to-\infty} l_{3s}(t) = \delta_2'$ and $l_3(0) = I^2_{\gamma_2}(w_2)$; and

- $l_4 : [0,\infty) \to M \setminus \partial M$ satisfies $dl_4/dt = -X_{f_3} \circ l_4$, and $l_4(0) = j_{p_3}(w_3)$ and $\lim_{t\to\infty} l_3(t) = p_3$.

We denote by $\mathcal{M}_N(\gamma_2, \delta_2')$ the set of such unparameterized negative gradient trajectories l_2, by $\mathcal{M}(p_1, \delta_2', p_3)$ the set of such gradient trees (l_1, l_3, l_4). Then, if $x_{2s} \in H_1^{\mu(\delta_1)+1} \cup \cdots \cup H_{k_N}^{\mu(\delta_N)+1} \subset e^2_{\gamma_2}$, we may identify the set of such $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon})$ with

$$\bigcup_{\delta_2'} \mathcal{M}_N(\gamma_2, \delta_2') \times \mathcal{M}(p_1, \delta_2', p_3).$$

Then we obtain

$$\begin{aligned} n(p_1, \gamma_2, p_3) = &\sum_{\gamma_1', \delta_3'} \sharp\mathcal{M}(p_1, \gamma_1') \sharp\mathcal{M}_N(\gamma_1', \gamma_2, \delta_3') \sharp\mathcal{M}(\delta_3', p_3) \\ &+ \sum_{\delta_2'} \sharp\mathcal{M}(\gamma_2, \delta_2') \sharp\mathcal{M}(p_1, \delta_2', p_3). \end{aligned}$$

How complicated they are! But we have to go ahead!

Fourthly, we fix $n(p_1, \gamma_2, \gamma_3)$. Let p_1 be an interior critical points of f_1, and γ_2, γ_3 positive boundary critical points of $f_{2\partial M}, f_{3\partial M}$, respectively. Suppose $\mu(p_1) + \mu(\gamma_2) - n = \mu(\gamma_3)$. For small $\varepsilon > 0$, we define

$$\begin{aligned} &I_\varepsilon(p_1, \gamma_2, \gamma_3) := \\ &\quad \left\{ (x_1, x_2, x_3) \in B^{\mu(p_1)} \times e^2_{\gamma_2} \times B^{n-\mu(\gamma_3)} : i_{p_1}(x_1) = \psi_\varepsilon \circ I^2_{\gamma_2}(x_2) = j_{\gamma_3}(x_3) \right\}. \end{aligned}$$

Since $\varepsilon > 0$ is small enough, for each $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon}) \in I_\varepsilon(p_1, \gamma_2, \gamma_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s := (x_{1s}, x_{2s}, x_{3s}) \in I_s(p_1, \gamma_2, \gamma_3)$. There are two possibilities: First $x_{2s} \in B^{\mu(\gamma_2)} \subset e^2_{\gamma_2}$, and secondly $x_{2s} \in H_1^{\mu(\delta_1)+1} \cup \cdots \cup H_{k_N}^{\mu(\delta_N)+1} \subset e^2_{\gamma_2}$. Suppose $x_{2s} \in B^{\mu(\gamma_2)}$, and then (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree (l_{1s}, l_{2s}, l_{3s}) such that

- $l_{1s} : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = p_1$ and $l_{1s}(0) = i_{p_1}(x_{1s})$;
- $l_{2s} : (-\infty, 0] \to \partial M$ satisfies $dl_{2s}/dt = -X_{f_{2\partial M}} \circ l_{2s}$, and $\lim_{t\to-\infty} l_{2s}(t) = \gamma_2$ and $l_{2s}(0) = I^2_{\gamma_2}(x_{2s})$; and
- $l_{3s} : [0,\infty) \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_3} \circ l_{3s}$, and $l_{3s}(0) = j_{\gamma_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = \gamma_3$.

We define $(w_1, w_2, w_3) := \lim_{s\to 0}(x_{1s}, x_{2s}, x_{3s})$. Then, when $s \to 0$, l_{1s} converges to a broken trajectory (l_0, l_1) such that

- l_0 is a negative gradient trajectory of f_1 from p_1 to γ_1', where γ_1' is a positive boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1') + 1 = \mu(p_1)$; and
- $l_1 : (-\infty, 0] \to \partial M$ satisfies $dl_1/dt = -X_{f_{1\partial M}} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = \gamma_1'$ and $l_1(0) = I^2_{\gamma_2}(w_2)$,

and (l_{2s}, l_{3s}) converges to (l_2, l_3) such that

- $l_2 : (-\infty, 0] \to \partial M$ satisfies $dl_2/dt = -X_{f_{2\partial M}} \circ l_2$, and $\lim_{t\to-\infty} l_2(t) = \gamma_2$ and $l_2(0) = I^2_{\gamma_2}(w_2)$; and
- $l_3 : [0,\infty) \to \partial M$ satisfies $dl_3/dt = -X_{f_{3\partial M}} \circ l_3$, and $l_3(0) = I^2_{\gamma_2}(w_2)$ and $\lim_{t\to\infty} l_3(t) = \gamma_3$.

We denote by $\mathcal{M}(p_1,\gamma_1')$ the set of such unparameterized negative gradient trajectories l_0, and $\mathcal{M}_N(\gamma_1',\gamma_2,\gamma_3)$ the set of such gradient trees (l_1,l_2,l_3). Then, if $x_{2\varepsilon}\in B^{\mu(\gamma_2)}\subset e^2_{\gamma_2}$, we may identify the set of such $(x_{1\varepsilon},x_{2\varepsilon},x_{3\varepsilon})$ with

$$\bigcup_{\gamma_1'}\mathcal{M}(p_1,\gamma_1')\times\mathcal{M}_N(\gamma_1',\gamma_2,\gamma_3).$$

Next, suppose $x_{2s}\in H_1^{\mu(\delta_1)+1}\cup\cdots\cup H_{k_N}^{\mu(\delta_N)+1}$, and then (x_{1s},x_{2s},x_{3s}) corresponds to a gradient tree $(l_{1s},l_{2s},l_{3s},l_{4s})$ such that

- $l_{1s}:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_{1s}/dt=-X_{f_1}\circ l_{1s}$, and $\lim_{t\to-\infty}l_{1s}(t)=p_1$ and $l_{1s}(0)=i_{p_1}(x_{1s})$;
- l_{2s} is a negative gradient trajectory of $f_{2\partial M}$ from γ_2 to δ_2', where δ_2' is a negative boundary critical point of $f_{2\partial M}$ with $\mu(\gamma_2)=\mu(\delta_2')+1$;
- $l_{3s}:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_{3s}/dt=-X_{f_2}\circ l_{3s}$, and $\lim_{t\to-\infty}l_{2s}(t)=\delta_2'$ and $l_{2s}(0)=I^2_{\gamma_2}(x_{2s})$; and
- $l_{4s}:[0,\infty)\to M\setminus\partial M$ satisfies $dl_{4s}/dt=-X_{f_3}\circ l_{4s}$, and $l_{4s}(0)=j_{\gamma_3}(x_{3s})$ and $\lim_{t\to\infty}l_{3s}(t)=\gamma_3$.

But, for our Morse functions, there is no broken trajectory (l_{3s},l_{4s}) from δ_2' to γ_3 since $f_{2\partial M}(\delta_2')<0, f_{3\partial M}(\gamma_3)>0$ and the values of f_2,f_3 must decrease along the broken trajectory, and this case does not occur. Note that we do not have such mechanism in Floer case, so the products on Floer homology would be more complicated, see Section 5! Then we obtain

$$n(p_1,\gamma_2,\gamma_3)=\sum_{\gamma_1'}\sharp\mathcal{M}(p_1,\gamma_1')\sharp\mathcal{M}_N(\gamma_1',\gamma_2,\gamma_3).$$

Fifthly, we fix $n(\gamma_1,p_2,p_3)$. Let p_2,p_3 be interior critical points of f_2,f_3, respectively, and γ_1 a positive boundary critical point of $f_{1\partial M}$. Suppose $\mu(\gamma_1)+\mu(p_2)-n=\mu(p_3)$. For small $\varepsilon>0$, we define

$$I_\varepsilon(\gamma_1,p_2,p_3):=\left\{(x_1,x_2,x_3)\in e^1_{\gamma_1}\times B^{\mu(p_2)}\times B^{n-\mu(p_3)}:I^1_{\gamma_1}(x_1)=\psi_\varepsilon\circ i_{p_2}(x_2)=j_{p_3}(x_3)\right\}.$$

Since $\varepsilon>0$ is small enough, for each $(x_{1\varepsilon},x_{2\varepsilon},x_{3\varepsilon})\in I_\varepsilon(\gamma_1,p_2,p_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s:=(x_{1s},x_{2s},x_{3s})\in I_s(\gamma_1,p_2,p_3)$. Since $U^{f_1}_{\gamma_1}\subset\partial M$ and $\psi_s(U^{f_2}_{p_2})\subset M\setminus\partial M$, $x_{1s}\in H_1^{\mu(\delta_1)+1}\setminus\partial H_1^{\mu(\delta_1)+1}\cup\cdots\cup H_{k_N}^{\mu(\delta_N)+1}\setminus\partial H_{k_N}^{\mu(\delta_N)+1}\subset e^1_{\gamma_1}$. Then (x_{1s},x_{2s},x_{3s}) corresponds to a gradient tree $(l_{0s},l_{1s},l_{2s},l_{3s})$ such that

- l_{0s} is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ_1', where δ_1' is a negative boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1)=\mu(\delta_1')+1$;
- $l_{1s}:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_{1s}/dt=-X_{f_1}\circ l_{1s}$, and $\lim_{t\to-\infty}l_{1s}(t)=\delta_1'$ and $l_{1s}(0)=I^1_{\gamma_1}(x_{1s})$;
- $l_{2s}:(-\infty,0]\to M\setminus\partial M$ satisfies $dl_{2s}/dt=-X_{f_2}\circ l_{2s}$, and $\lim_{t\to-\infty}l_{2s}(t)=p_2$ and $l_{2s}(0)=i_{p_2}(x_{2s})$; and
- $l_{3s}:[0,\infty)\to M\setminus\partial M$ satisfies $dl_{3s}/dt=-X_{f_3}\circ l_{3s}$, and $l_{3s}(0)=j_{p_3}(x_{3s})$ and $\lim_{t\to\infty}l_{3s}(t)=p_3$.

We define $(w_1, w_2, w_3) := \lim_{s\to 0}(x_{1s}, x_{2s}, x_{3s})$. Then, when $s \to 0$, $(l_{0s}, l_{1s}, l_{2s}, l_{3s})$ converges to (l_0, l_1, l_2, l_3) such that

- l_0 is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ_1';
- $l_1 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_1/dt = -X_{f_1} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = \delta_1'$ and $l_1(0) = I^1_{\gamma_1}(w_1)$;
- $l_2 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_2/dt = -X_{f_2} \circ l_2$, and $\lim_{t\to-\infty} l_2(t) = p_2$ and $l_2(0) = i_{p_2}(w_2)$; and
- $l_3 : [0, \infty) \to M \setminus \partial M$ satisfies $dl_3/dt = -X_{f_3} \circ l_3$, and $l_3(0) = j_{p_3}(w_3)$ and $\lim_{t\to\infty} l_3(t) = p_3$.

We denote by $\mathcal{M}_N(\gamma_1, \delta_1')$ the set of such unparameterized negative gradient trajectories l_0, $\mathcal{M}(\delta_1', p_2, p_3)$ the set of such gradient trees (l_1, l_2, l_3). Then we obtain

$$n(\gamma_1, p_2, p_3) = \sum_{\delta_1'} \sharp\mathcal{M}_N(\gamma_1, \delta_1')\sharp\mathcal{M}(\delta_1', p_2, p_3).$$

Sixthly, we fix $n(\gamma_1, p_2, \gamma_3)$. Let p_2 be an interior critical points of f_2, and γ_1, γ_3 positive boundary critical points of $f_{1\partial M}, f_{3\partial M}$, respectively. Suppose $\mu(\gamma_1) + \mu(p_2) - n = \mu(\gamma_3)$. For small $\varepsilon > 0$, we define

$$I_\varepsilon(\gamma_1, p_2, \gamma_3) := \left\{(x_1, x_2, x_3) \in e^1_{\gamma_1} \times B^{\mu(p_2)} \times B^{n-\mu(\gamma_3)} : I^1_{\gamma_1}(x_1) = \psi_\varepsilon \circ i_{p_2}(x_2) = j_{\gamma_3}(x_3)\right\}.$$

Since $\varepsilon > 0$ is small enough, for each $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon}) \in I_\varepsilon(\gamma_1, p_2, \gamma_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s := (x_{1s}, x_{2s}, x_{3s}) \in I_s(\gamma_1, p_2, \gamma_3)$. Since $U^{f_1}_{\gamma_1} \subset \partial M$ and $\psi_s(U^{f_2}_{p_2}) \subset M \setminus \partial M$, $x_{1s} \in H_1^{\mu(\delta_1)+1} \setminus \partial H_1^{\mu(\delta_1)+1} \cup \cdots \cup H_{k_N}^{\mu(\delta_N)+1} \setminus \partial H_{k_N}^{\mu(\delta_N)+1} \subset e^1_{\gamma_1}$. Then (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree $(l_{0s}, l_{1s}, l_{2s}, l_{3s})$ such that

- l_{0s} is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ_1', where δ_1' is a negative boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1) = \mu(\delta_1') + 1$;
- $l_{1s} : (-\infty, 0] \to M\setminus\partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = \delta_1'$ and $l_{1s}(0) = I^1_{\gamma_1}(x_{1s})$;
- $l_{2s} : (-\infty, 0] \to M\setminus\partial M$ satisfies $dl_{2s}/dt = -X_{f_2} \circ l_{2s}$, and $\lim_{t\to-\infty} l_{2s}(t) = p_2$ and $l_{2s}(0) = i_{p_2}(x_{2s})$; and
- $l_{3s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_3} \circ l_{3s}$, and $l_{3s}(0) = j_{\gamma_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = \gamma_3$.

But, for our Morse functions, there is no broken trajectory (l_{1s}, l_{3s}) from δ_1' to γ_3 since $f_{1\partial M}(\delta_1') < 0, f_{3\partial M}(\gamma_3) > 0$ and the values of f_1, f_3 must decrease along the broken trajectory, and this case does not occur. Then we obtain

$$n(\gamma_1, p_2, \gamma_3) = 0.$$

Seventhly, we fix $n(\gamma_1, \gamma_2, p_3)$. Let p_3 be an interior critical points of f_3, and γ_1, γ_2 positive boundary critical points of $f_{1\partial M}, f_{2\partial M}$, respectively. Suppose $\mu(\gamma_1)+\mu(\gamma_2) - n = \mu(p_3)$. For small $\varepsilon > 0$, we define

$$I_\varepsilon(\gamma_1, \gamma_2, p_3) := \left\{(x_1, x_2, x_3) \in e^1_{\gamma_1} \times e^2_{\gamma_2} \times B^{n-\mu(p_3)} : I^1_{\gamma_1}(x_1) = \psi_\varepsilon \circ I^2_{\gamma_2}(x_2) = j_{p_3}(x_3)\right\}.$$

Since $\varepsilon > 0$ is small enough, for each $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon}) \in I_\varepsilon(\gamma_1, \gamma_2, p_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s := (x_{1s}, x_{2s}, x_{3s}) \in I_s(\gamma_1, \gamma_2, p_3)$. Since $U^{f_1}_{\gamma_1} \subset \partial M$ and $\psi_\varepsilon(U^{f_1}_{\gamma_2}) \subset M \setminus \partial M$, $x_{1s} \in H^{\mu(\delta_1)+1}_1 \setminus \partial H^{\mu(\delta_1)+1}_1 \cup \cdots \cup H^{\mu(\delta_N)+1}_{k_N} \setminus \partial H^{\mu(\delta_N)+1}_{k_N} \subset e^1_{\gamma_1}$. There are two possibilities: First $x_{2s} \in B^{\mu(\gamma_2)} \subset e^2_{\gamma_2}$, and secondly $x_{2s} \in H^{\mu(\delta'_1)+1}_1 \cup \cdots \cup H^{\mu(\delta'_{N'})+1}_{k_{N'}} \subset e^2_{\gamma_2}$. Suppose $x_{2s} \in B^{\mu(\gamma_2)}$, and then (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree $(l_{0s}, l_{1s}, l_{2s}, l_{3s})$ such that

- l_{0s} is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ''_1, where δ''_1 is a negative boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1) = \mu(\delta''_1) + 1$;
- $l_{1s} : (-\infty, 0] \to M\setminus\partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = \delta''_1$ and $l_{1s}(0) = I^1_{\gamma_1}(x_{1s})$;
- $l_{2s} : (-\infty, 0] \to \partial M$ satisfies $dl_{2s}/dt = -X_{f_{2\partial M}} \circ l_{2s}$, and $\lim_{t\to-\infty} l_{2s}(t) = \gamma_2$ and $l_{2s}(0) = I^2_{\gamma_2}(x_{2s})$; and
- $l_{3s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_3} \circ l_{3s}$, and $l_{3s}(0) = j_{p_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = p_3$.

We define $(w_1, w_2, w_3) := \lim_{s\to 0}(x_{1s}, x_{2s}, x_{3s})$. Then, when $s \to 0$, (l_{0s}, l_{1s}, l_{2s}) converges to (l_0, l_1, l_2) such that

- l_0 is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ''_1;
- $l_1 : (-\infty, 0] \to \partial M$ satisfies $dl_1/dt = -X_{f_{1\partial M}} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = \delta''_1$ and $l_1(0) = I^1_{\gamma_1}(w_1)$; and
- $l_2 : (-\infty, 0] \to \partial M$ satisfies $dl_2/dt = -X_{f_{2\partial M}} \circ l_2$, and $\lim_{t\to-\infty} l_2(t) = \gamma_2$ and $l_2(0) = I^2_{\gamma_2}(w_2)$,

and l_{3s} converges to a broken trajectory (l_3, l_4) such that

- $l_3 : [0, \infty) \to \partial M$ satisfies $dl_3/dt = -X_{f_{3\partial M}} \circ l_3$, and $l_3(0) = I^2_{\gamma_2}(w_2)$ and $\lim_{t\to\infty} l_3(t) = \delta''_3$, where δ''_3 is a negative boundary critical point of $f_{3\partial M}$ with $\mu(\delta''_3) = \mu(p_3)$; and
- l_4 is a negative gradient trajectory of f_3 from δ''_3 to p_3.

We denote by $\mathcal{M}_N(\gamma_1, \delta''_1)$ the set of such unparameterized negative gradient trajectories l_0, $\mathcal{M}_N(\delta''_1, \gamma_2, \delta''_3)$ the set of such gradient trees (l_1, l_2, l_3), and $\mathcal{M}(\delta''_3, p_3)$ the set of such unparameterized negative gradient trajectories l_4. Then, if $x_{2\varepsilon} \in B^{\mu(\gamma_2)} \subset e^2_{\gamma_2}$, we may identify the set of such $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon})$ with

$$\bigcup_{\delta''_1, \delta''_3} \mathcal{M}_N(\gamma_1, \delta''_1) \times \mathcal{M}_N(\delta''_1, \gamma_2, \delta''_3) \times \mathcal{M}(\delta''_3, p_3).$$

Next, suppose $x_{2s} \in H^{\mu(\delta'_1)+1}_1 \cup \cdots \cup H^{\mu(\delta'_{N'})+1}_{k_{N'}}$, and then (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree $(l_{1s}, l_{2s}, l_{3s}, l_{4s})$ such that

- l_{0s} is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ''_1, where δ''_1 is a negative boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1) = \mu(\delta''_1) + 1$;
- $l_{1s} : (-\infty, 0] \to M\setminus\partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = \delta''_1$ and $l_{1s}(0) = I^1_{\gamma_1}(x_{1s})$;
- l_{2s} is a negative gradient trajectory of $f_{2\partial M}$ from γ_2 to δ''_2, where δ''_2 is a negative boundary critical point of $f_{2\partial M}$ with $\mu(\gamma_2) = \mu(\delta''_2) + 1$;

- $l_{3s} : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_2} \circ l_{3s}$, and $\lim_{t\to-\infty} l_{3s}(t) = \delta_2''$ and $l_{3s}(0) = I^2_{\gamma_2}(x_{2s})$; and
- $l_{4s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{4s}/dt = -X_{f_3} \circ l_{4s}$, and $l_{4s}(0) = j_{p_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = p_3$.

We define $(w_1, w_2, w_3) := \lim_{s\to 0}(x_{1s}, x_{2s}, x_{3s})$. Then, when $s \to 0$, $(l_{0s}, l_{1s}, l_{2s}, l_{3s}, l_{4s})$ converges to $(l_0, l_1, l_2, l_3, l_4)$ such that

- l_0 is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ_1'';
- $l_1 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_1/dt = -X_{f_1} \circ l_1$, and $\lim_{t\to-\infty} l_1(t) = \delta_1''$ and $l_1(0) = I^1_{\gamma_1}(w_1)$;
- l_2 is a negative gradient trajectory of $f_{2\partial M}$ from γ_2 to δ_2'';
- $l_3 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_3/dt = -X_{f_2} \circ l_3$, and $\lim_{t\to-\infty} l_3(t) = \delta_2''$ and $l_3(0) = I^2_{\gamma_2}(w_2)$; and
- $l_4 : [0, \infty) \to M \setminus \partial M$ satisfies $dl_4/dt = -X_{f_3} \circ l_4$, and $l_4(0) = j_{p_3}(w_3)$ and $\lim_{t\to\infty} l_3(t) = p_3$.

We denote by $\mathcal{M}_N(\gamma_1, \delta_1'')$ the set of such unparameterized negative gradient trajectories l_0, $\mathcal{M}_N(\gamma_2, \delta_2'')$ the set of such unparameterized negative gradient trajectories l_2, $\mathcal{M}(\delta_1'', \delta_2'', p_3)$ the set of such gradient trees (l_1, l_3, l_4). Then, if $x_{2s} \in H_1^{\mu(\delta_1')+1} \cup \cdots \cup H_{k_{N'}}^{\mu(\delta_{N'}')+1} \subset e^2_{\gamma_2}$, we may identify the set of such $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon})$ with

$$\bigcup_{\delta_1'',\delta_2''} \mathcal{M}_N(\gamma_1, \delta_1'') \times \mathcal{M}_N(\gamma_2, \delta_2'') \times \mathcal{M}(\delta_1'', \delta_2'', p_3).$$

Then we obtain

$$\begin{aligned} n(\gamma_1, \gamma_2, p_3) = &\sum_{\delta_1'',\delta_3''} \sharp\mathcal{M}_N(\gamma_1, \delta_1'')\sharp\mathcal{M}_N(\delta_1'', \gamma_2, \delta_3'')\sharp\mathcal{M}(\delta_3'', p_3) \\ &+ \sum_{\delta_1'',\delta_2''} \sharp\mathcal{M}_N(\gamma_1, \delta_1'')\sharp\mathcal{M}_N(\gamma_2, \delta_2'')\sharp\mathcal{M}(\delta_1'', \delta_2'', p_3). \end{aligned}$$

Finally, we fix $n(\gamma_1, \gamma_2, \gamma_3)$ at last! Let γ_1, γ_2 and γ_3 positive boundary critical points of $f_{1\partial M}, f_{2\partial M}$ and $f_{3\partial M}$, respectively. Suppose $\mu(\gamma_1) + \mu(\gamma_2) - n = \mu(\gamma_3)$. For small $\varepsilon > 0$, we define

$$\begin{aligned} &I_\varepsilon(\gamma_1, \gamma_2, \gamma_3) := \\ &\quad \left\{(x_1, x_2, x_3) \in e^1_{\gamma_1} \times e^2_{\gamma_2} \times B^{n-\mu(\gamma_3)} : I^1_{\gamma_1}(x_1) = \psi_\varepsilon \circ I^2_{\gamma_2}(x_2) = j_{\gamma_3}(x_3)\right\}. \end{aligned}$$

Since $\varepsilon > 0$ is small enough, for each $(x_{1\varepsilon}, x_{2\varepsilon}, x_{3\varepsilon}) \in I_\varepsilon(\gamma_1, \gamma_2, \gamma_3)$, we may find a smooth family $\{x_s\}_{s\in(0,\varepsilon]}$ such that $x_s := (x_{1s}, x_{2s}, x_{3s}) \in I_s(\gamma_1, \gamma_2, \gamma_3)$. Since $U^{f_1}_{\gamma_1} \subset \partial M$ and $\psi_\varepsilon(U^{f_1}_{\gamma_2}) \subset M \setminus \partial M$, $x_{1s} \in H_1^{\mu(\delta_1)+1} \setminus \partial H_1^{\mu(\delta_1)+1} \cup \cdots \cup H_{k_N}^{\mu(\delta_N)+1} \setminus \partial H_{k_N}^{\mu(\delta_N)+1} \subset e^1_{\gamma_1}$. There are two possibilities: First $x_{2s} \in B^{\mu(\gamma_2)} \subset e^2_{\gamma_2}$, and secondly $x_{2s} \in H_1^{\mu(\delta_1')+1} \cup \cdots \cup H_{k_{N'}}^{\mu(\delta_{N'}')+1} \subset e^2_{\gamma_2}$. Suppose $x_{2s} \in B^{\mu(\gamma_2)}$, and then (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree $(l_{0s}, l_{1s}, l_{2s}, l_{3s})$ such that

- l_{0s} is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ_1'', where δ_1'' is a negative boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1) = \mu(\delta_1'') + 1$;
- $l_{1s} : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = \delta_1''$ and $l_{1s}(0) = I^1_{\gamma_1}(x_{1s})$;

- $l_{2s} : (-\infty, 0] \to \partial M$ satisfies $dl_{2s}/dt = -X_{f_{2\partial M}} \circ l_{2s}$, and $\lim_{t\to-\infty} l_{2s}(t) = \gamma_2$ and $l_{2s}(0) = I^2_{\gamma_2}(x_{2s})$; and
- $l_{3s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{3s}/dt = -X_{f_3} \circ l_{3s}$, and $l_{3s}(0) = j_{p_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = \gamma_3$.

But, for our Morse functions, there is no broken trajectory (l_{1s}, l_{2s}) from δ''_1 to γ_3 since $f_{1\partial M}(\delta''_1) < 0, f_{3\partial M}(\gamma_3) > 0$ and the values of f_1, f_3 must decrease along the broken trajectory, and this case does not occur. Next, suppose $x_{2s} \in H_1^{\mu(\delta'_1)+1} \cup \cdots \cup H_{k_{N'}}^{\mu(\delta'_{N'})+1}$, and then (x_{1s}, x_{2s}, x_{3s}) corresponds to a gradient tree $(l_{1s}, l_{2s}, l_{3s}, l_{4s})$ such that

- l_{0s} is a negative gradient trajectory of $f_{1\partial M}$ from γ_1 to δ''_1, where δ''_1 is a negative boundary critical point of $f_{1\partial M}$ with $\mu(\gamma_1) = \mu(\delta''_1) + 1$;
- $l_{1s} : (-\infty, 0] \to M\setminus\partial M$ satisfies $dl_{1s}/dt = -X_{f_1} \circ l_{1s}$, and $\lim_{t\to-\infty} l_{1s}(t) = \delta''_1$ and $l_{1s}(0) = I^1_{\gamma_1}(x_{1s})$;
- l_{2s} is a negative gradient trajectory of $f_{2\partial M}$ from γ_2 to δ''_2, where δ''_2 is a negative boundary critical point of $f_{2\partial M}$ with $\mu(\gamma_2) = \mu(\delta''_2) + 1$;
- $l_{3s} : (-\infty, 0] \to M\setminus\partial M$ satisfies $dl_{3s}/dt = -X_{f_2} \circ l_{3s}$, and $\lim_{t\to-\infty} l_{3s}(t) = \delta''_2$ and $l_{3s}(0) = I^2_{\gamma_2}(x_{2s})$; and
- $l_{4s} : [0, \infty) \to M \setminus \partial M$ satisfies $dl_{4s}/dt = -X_{f_3} \circ l_{4s}$, and $l_{4s}(0) = j_{p_3}(x_{3s})$ and $\lim_{t\to\infty} l_{3s}(t) = \gamma_3$.

But, for our Morse functions, there is no gradient tree (l_{1s}, l_{3s}, l_{4s}) from δ''_1, δ''_2 to γ_3 since $f_{1\partial M}(\delta''_1) < 0, f_{2\partial M}(\delta''_2) < 0, f_{3\partial M}(\gamma_3) > 0$ and the values of f_1, f_2, f_3 must decrease along the gradient tree, and this case does not occur. Then we obtain

$$n(\gamma_1, \gamma_2, \gamma_3) = 0.$$

Now we redefine the linear map $m_2 : C_{k_1}(f_1) \otimes C_{k_2}(f_2) \to C_{k_1+k_2-n}(f_3)$ by

$$\begin{aligned}
m_2(p_1, p_2) &:= \sum_{p_3} \sharp\mathcal{M}(p_1, p_2, p_3)p_3 + \sum_{\gamma_3} \sharp\mathcal{M}(p_1, p_2, \gamma_3)\gamma_3,\\
m_2(p_1, \gamma_2) &:= \sum_{\gamma'_1, \delta'_3, p_3} \sharp\mathcal{M}(p_1, \gamma'_1)\sharp\mathcal{M}_N(\gamma'_1, \gamma_2, \delta'_3)\sharp\mathcal{M}(\delta'_3, p_3)p_3\\
&\quad + \sum_{\delta'_2, p_3} \sharp\mathcal{M}_N(\gamma_2, \delta'_2)\sharp\mathcal{M}(p_1, \delta'_2, p_3)p_3\\
&\quad + \sum_{\gamma'_1, \gamma_3} \sharp\mathcal{M}(p_1, \gamma'_1)\sharp\mathcal{M}_N(\gamma'_1, \gamma_2, \gamma_3)\gamma_3,\\
m_2(\gamma_1, p_2) &:= \sum_{\delta'_1, p_3} \sharp\mathcal{M}_N(\gamma_1, \delta'_1)\sharp\mathcal{M}(\delta'_1, p_2, p_3)p_3,\\
m_2(\gamma_1, \gamma_2) &:= \sum_{\delta''_1, \delta''_3, p_3} \sharp\mathcal{M}_N(\gamma_1, \delta''_1)\sharp\mathcal{M}_N(\delta''_1, \gamma_2, \delta''_3)\sharp\mathcal{M}(\delta''_3, p_3)p_3\\
&\quad + \sum_{\delta''_1, \delta''_2, p_3} \sharp\mathcal{M}_N(\gamma_1, \delta''_1)\sharp\mathcal{M}_N(\gamma_2, \delta''_2)\sharp\mathcal{M}(\delta''_1, \delta''_2, p_3)p_3.
\end{aligned}$$

Note that the dimension of each moduli space above is 0. Then we obtain the following theorem from Theorem 3.2:

Theorem 3.3. *We denote by $\partial^{f_1}, \partial^{f_2}$ and ∂^{f_3} the boundary operators of Morse complex for f_1, f_2 and f_3, respectively. Then we obtain the Leibniz rule: (We omit the sign convention.)*

$$\partial^{f_3} m_2(*_1 \otimes *_2) = m_2(\partial^{f_1} *_1 \otimes *_2) \pm m_2(*_1 \otimes \partial^{f_2} *_2),$$

where $*_i$ is an interior critical point of f_i or a positive boundary critical point of $f_{i\partial M}$, for $i = 1, 2$.

Note that we may also prove this theorem by observing the boundary of 1-dimensional moduli spaces of gradient trees, see Section 4, which is very important for product structures on Floer homology of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end, see Section 5.

4. Gradient trees

In this section, we prove the Leibniz rules on Morse homology of manifolds with boundary in terms of gradient trees. But, before the Leibniz rules, we briefly review the proof of $\partial_{k-1} \circ \partial_k = 0$ in terms of gradient trajectories, see [1].

First we recall our settings. Let M be an n-dimensional oriented compact manifold with boundary ∂M. We identify a collar neighborhood of the boundary with $[0, 1) \times \partial M$, and denote by r the standard coordinate on the first factor. Take a Riemannian metric g on $M \setminus \partial M$ such that $g|_{(0,1)\times\partial M} = \frac{1}{r} dr \otimes dr + r g_{\partial M}$, where $g_{\partial M}$ is a Riemannian metric on ∂M. Let f be a Morse function on $M \setminus \partial M$ which satisfies the following conditions:

- There is a Morse function $f_{\partial M}$ on ∂M such that $f|_{(0,1)\times\partial M} = r f_{\partial M}$; and
- If γ is a critical point of $f_{\partial M}$, then $f_{\partial M}(\gamma)$ is not equal to zero.

We call $\gamma \in \partial M$ a *positive boundary critical point* if γ is a critical point of $f_{\partial M}$ and $f_{\partial M}(\gamma) > 0$, and similarly, we call $\delta \in \partial M$ a *negative boundary critical point* if δ is a critical point of $f_{\partial M}$ and $f_{\partial M}(\delta) < 0$. On the other hand, we call $p \in M \setminus \partial M$ an *interior critical point* if p is a critical point of f. Note that we always use notation $\gamma, \gamma', \gamma'' \in \partial M$ for positive boundary critical points, $\delta, \delta', \delta'' \in \partial M$ for negative boundary critical points, and $p, p', p'' \in M \setminus \partial M$ for interior critical points. On the collar neighborhood $(0, 1) \times \partial M$, the gradient vector field X_f with respect to f and g is $r f_{\partial M} \frac{\partial}{\partial r} + X_{f_{\partial M}}$, where $X_{f_{\partial M}}$ is the gradient vector field with respect to $f_{\partial M}$ and $g_{\partial M}$, and we define a vector field $\overline{X}_f$ on M by

$$\overline{X}_f := \begin{cases} X_f, & \text{on } M \setminus \partial M, \\ X_{f_{\partial M}}, & \text{on } \{0\} \times \partial M. \end{cases}$$

We define the moduli spaces of gradient trajectories. Let p, p' be interior critical points of f. We denote by $\widetilde{\mathcal{M}}(p, p')$ the set of maps $l : \mathbb{R} \to M \setminus \partial M$ such that

- $\dfrac{\partial l}{\partial t} = -\overline{X}_f$; and
- $\lim\limits_{t\to-\infty} l(t) = p$ and $\lim\limits_{t\to\infty} l(t) = p'$.

We define an equivalence relation $l \sim l'$ if $l(t) = l'(t + c)$, for some $c \in \mathbb{R}$, and we denote by $\mathcal{M}(p, p')$ the set of the equivalence classes. Similarly, we define $\mathcal{M}(p, \gamma)$ for an interior critical point p of f and a positive boundary critical point γ of $f_{\partial M}$, and $\mathcal{M}(\delta, p)$ for a negative boundary critical point δ of $f_{\partial M}$ and an interior critical point p of f. Let γ, γ' be positive boundary critical points of $f_{\partial M}$. We denote by $\widetilde{\mathcal{M}}_N(\gamma, \gamma')$ the set of maps $l : \mathbb{R} \to \partial M$ such that

- $\dfrac{\partial l}{\partial t} = -\overline{X}_f$; and
- $\lim_{t\to-\infty} l(t) = \gamma$ and $\lim_{t\to\infty} l(t) = \gamma'$.

We define an equivalence relation $l \sim l'$ if $l(t) = l'(t+c)$, for some $c \in \mathbb{R}$, and we denote by $\mathcal{M}_N(\gamma,\gamma')$ the set of the equivalence classes. Similarly, we define $\mathcal{M}_N(\gamma,\delta)$ for a positive boundary critical point γ of $f_{\partial M}$ and a negative boundary critical point δ of $f_{\partial M}$, and $\mathcal{M}_N(\delta,\delta')$ for negative boundary critical points δ,δ' of $f_{\partial M}$. Note that, since there is no negative gradient trajectories from a negative boundary critical point δ to a positive boundary critical point γ, $\mathcal{M}_N(\delta,\gamma) = \emptyset$. Then we have the following theorem, see [1]:

Theorem 4.1. *We may take a generic f so that the following hold:*

(a) *$\mathcal{M}(p,p')$ is an orientable smooth manifold of dimension $\mu(p)-\mu(p')-1$. If $\dim\mathcal{M}(p,p') = 0$, then $\mathcal{M}(p,p')$ is compact. If $\dim\mathcal{M}(p,p') = 1$, then $\mathcal{M}(p,p')$ can be compactified into $\overline{\mathcal{M}}(p,p')$, whose boundary is*

$$\begin{aligned}\partial\overline{\mathcal{M}}(p,p') = &\bigcup_{\mu(p'')=\mu(p)-1} \mathcal{M}(p,p'')\times\mathcal{M}(p'',p')\\ &\cup \bigcup_{\substack{\mu(\gamma)=\mu(p)-1\\ \mu(\delta)=\mu(\gamma)-1}} \mathcal{M}(p,\gamma)\times\mathcal{M}_N(\gamma,\delta)\times\mathcal{M}(\delta,p'),\end{aligned}$$

where p'' is an interior critical point, γ is a positive boundary critical point, and δ is a negative boundary critical point.

(b) *$\mathcal{M}(p,\gamma)$ is an orientable smooth manifold of dimension $\mu(p)-\mu(\gamma)-1$. If $\dim\mathcal{M}(p,\gamma) = 0$, then $\mathcal{M}(p,\gamma)$ is compact. If $\dim\mathcal{M}(p,\gamma) = 1$, then $\mathcal{M}(p,\gamma)$ can be compactified into $\overline{\mathcal{M}}(p,\gamma)$, whose boundary is*

$$\partial\overline{\mathcal{M}}(p,\gamma) = \bigcup_{\mu(p')=\mu(p)-1} \mathcal{M}(p,p')\times\mathcal{M}(p',\gamma) \cup \bigcup_{\mu(\gamma')=\mu(\gamma)-1} \mathcal{M}(p,\gamma')\times\mathcal{M}_N(\gamma',\gamma),$$

where p' is an interior critical point, and γ' is a positive boundary critical point.

(c) *$\mathcal{M}(\delta,p)$ is an orientable smooth manifold of dimension $\mu(\delta)-\mu(p)$. If $\dim\mathcal{M}(\delta,p) = 0$, then $\mathcal{M}(\delta,p)$ is compact. If $\dim\mathcal{M}(\delta,p) = 1$, then $\mathcal{M}(\delta,p)$ can be compactified into $\overline{\mathcal{M}}(\delta,p)$, whose boundary is*

$$\partial\overline{\mathcal{M}}(\delta,p) = \bigcup_{\mu(p')=\mu(\delta)} \mathcal{M}(\delta,p')\times\mathcal{M}(p',p) \cup \bigcup_{\mu(\delta')=\mu(\delta)-1} \mathcal{M}_N(\delta,\delta')\times\mathcal{M}(\delta',p),$$

where p' is an interior critical point, and δ' is a negative boundary critical point.

(d) *$\mathcal{M}_N(\gamma,\gamma')$ is an orientable smooth manifold of dimension $\mu(\gamma)-\mu(\gamma')-1$. If $\dim\mathcal{M}_N(\gamma,\gamma') = 0$, then $\mathcal{M}_N(\gamma,\gamma')$ is compact. If $\dim\mathcal{M}_N(\gamma,\gamma') = 1$, then $\mathcal{M}_N(\gamma,\gamma')$ can be compactified into $\overline{\mathcal{M}}_N(\gamma,\gamma')$, whose boundary is*

$$\partial\overline{\mathcal{M}}_N(\gamma,\gamma') = \bigcup_{\mu(\gamma'')=\mu(\gamma)-1} \mathcal{M}_N(\gamma,\gamma'')\times\mathcal{M}_N(\gamma'',\gamma'),$$

where γ'' is a positive boundary critical point.

(e) *$\mathcal{M}_N(\gamma,\delta)$ is an orientable smooth manifold of dimension $\mu(\gamma)-\mu(\delta)-1$. If $\dim\mathcal{M}_N(\gamma,\delta) = 0$, then $\mathcal{M}_N(\gamma,\delta)$ is compact. If $\dim\mathcal{M}_N(\gamma,\delta) = 1$, then $\mathcal{M}_N(\gamma,\delta)$ can be compactified into $\overline{\mathcal{M}}_N(\gamma,\delta)$, whose boundary is*

$$\partial\overline{\mathcal{M}}_N(\gamma,\delta) = \bigcup_{\mu(\gamma')=\mu(\gamma)-1} \mathcal{M}_N(\gamma,\gamma')\times\mathcal{M}_N(\gamma',\delta)\cup \bigcup_{\mu(\delta')=\mu(\gamma)-1} \mathcal{M}_N(\gamma,\delta')\times\mathcal{M}_N(\delta',\delta),$$

where γ' is a positive boundary critical point and δ' is a negative boundary critical point.

Note that we put the orientation on moduli spaces which comes from the intersection number of $U_*^*, I_*^* : e_*^* \to M$ and S_*^*.

We may list every boundary components of 1-dimensional moduli spaces in Theorem 4.1 without omission by chasing critical points so that we obtain 1-dimensional moduli spaces after gluing gradient trajectories. Note that there is no broken negative trajectory from a negative boundary critical point to a positive boundary critical point.

We also have similar arguments for $\mathcal{M}_N(\delta_1, \delta_2)$, which we need for Morse complex of $f_{\partial M}$, but we do not use $\mathcal{M}_N(\delta_1, \delta_2)$ in this paper, see [1].

Remember that we defined

$$C_k(f) := \bigoplus_{\mu(p)=k} \mathbb{Z}p \oplus \bigoplus_{\mu(\gamma)=k} \mathbb{Z}\gamma,$$

and the linear map $\partial_k : C_k(f) \to C_{k-1}(f)$ by

$$\begin{aligned} \partial_k p &:= \sum_{\mu(p')=k-1} \sharp\mathcal{M}(p,p')p' + \sum_{\mu(\gamma')=k-1} \sharp\mathcal{M}(p,\gamma')\gamma', \\ \partial_k \gamma &:= \sum_{\mu(\gamma')=k-1} \sharp\mathcal{M}_N(\gamma,\gamma')\gamma' + \sum_{\substack{\mu(\delta)=k-1 \\ \mu(p')=k-1}} \sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,p')p'. \end{aligned}$$

We already proved the following theorem in Section 2 by considering the connecting homomorphisms. Here we prove the theorem by observing the boundary of 1-dimensional moduli spaces of unparameterized gradient trajectories.

Theorem 4.2. $\partial_{k-1} \circ \partial_k = 0$.

Proof. First we prove $\partial_{k-1} \circ \partial_k p = 0$, for an interior critical point p.

$$
\begin{aligned}
&\partial_{k-1} \circ \partial_k p \\
&= \partial_{k-1} \left\{ \sum_{p'} \sharp\mathcal{M}(p,p')p' + \sum_{\gamma'} \sharp\mathcal{M}(p,\gamma')\gamma' \right\} \\
&= \sum_{p'} \sharp\mathcal{M}(p,p') \left\{ \sum_{p''} \sharp\mathcal{M}(p',p'')p'' + \sum_{\gamma''} \sharp\mathcal{M}(p',\gamma'')\gamma'' \right\} \\
&\quad + \sum_{\gamma'} \sharp\mathcal{M}(p,\gamma') \left\{ \sum_{\gamma''} \sharp\mathcal{M}_N(\gamma',\gamma'')\gamma'' + \sum_{\delta,p''} \sharp\mathcal{M}_N(\gamma',\delta)\sharp\mathcal{M}(\delta,p'')p'' \right\} \\
&= \sum_{p''} \left\{ \sum_{p'} \sharp\mathcal{M}(p,p')\sharp\mathcal{M}(p',p'') + \sum_{\gamma',\delta} \sharp\mathcal{M}(p,\gamma')\sharp\mathcal{M}_N(\gamma',\delta)\sharp\mathcal{M}(\delta,p'') \right\} p'' \\
&\quad + \sum_{\gamma''} \left\{ \sum_{p'} \sharp\mathcal{M}(p,p')\sharp\mathcal{M}(p',\gamma'') + \sum_{\gamma'} \sharp\mathcal{M}(p,\gamma')\sharp\mathcal{M}_N(\gamma',\gamma'') \right\} \gamma'' \\
&\overset{\textbf{(a)(b)}}{=} \sum_{p''} \sharp\partial\overline{\mathcal{M}}(p,p'')p'' + \sum_{\gamma''} \sharp\partial\overline{\mathcal{M}}(p,\gamma'')\gamma'' \\
&= 0.
\end{aligned}
$$

Note that we use Theorem 4.1 **(a)** and **(b)** at $\overset{\textbf{(a)(b)}}{=}$. Hence $\partial_{k-1} \circ \partial_k p = 0$.

Next we prove $\partial_{k-1} \circ \partial_k \gamma = 0$, for a positive boundary critical point γ.

$$
\begin{aligned}
&\partial_{k-1} \circ \partial_k \gamma \\
&= \partial_{k-1} \left\{ \sum_{\gamma'} \sharp\mathcal{M}_N(\gamma,\gamma')\gamma' + \sum_{\delta,p'} \sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,p')p' \right\} \\
&= \sum_{\gamma'} \sharp\mathcal{M}_N(\gamma,\gamma') \left\{ \sum_{\gamma''} \sharp\mathcal{M}_N(\gamma',\gamma'')\gamma'' + \sum_{\delta,p''} \sharp\mathcal{M}_N(\gamma',\delta)\sharp\mathcal{M}(\delta,p'')p'' \right\} \\
&\quad + \sum_{\delta,p'} \sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,p') \left\{ \sum_{p''} \sharp\mathcal{M}(p',p'')p'' + \sum_{\gamma''} \sharp\mathcal{M}(p',\gamma'')\gamma'' \right\} \\
&= \sum_{p''} \left\{ \sum_{\gamma',\delta} \sharp\mathcal{M}_N(\gamma,\gamma')\sharp\mathcal{M}_N(\gamma',\delta)\sharp\mathcal{M}(\delta,p'') + \sum_{\delta,p'} \sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,p')\sharp\mathcal{M}(p',p'') \right\} p'' \\
&\quad + \sum_{\gamma''} \left\{ \sum_{\gamma'} \sharp\mathcal{M}_N(\gamma,\gamma')\sharp\mathcal{M}_N(\gamma',\gamma'') + \sum_{\delta,p'} \sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,p')\sharp\mathcal{M}(p',\gamma'') \right\} \gamma''.
\end{aligned}
$$

We define $n(\gamma,p'')$ and $n(\gamma,\gamma'')$ by

$$
\partial_{k-1} \circ \partial_k \gamma = \sum_{p''} n(\gamma,p'')p'' + \sum_{\gamma''} n(\gamma,\gamma'')\gamma''.
$$

Then

$$\begin{aligned}
n(\gamma, p'') &= \sum_{\gamma', \delta} \sharp\mathcal{M}_N(\gamma, \gamma')\sharp\mathcal{M}_N(\gamma', \delta)\sharp\mathcal{M}(\delta, p'') + \sum_{\delta, p'} \sharp\mathcal{M}_N(\gamma, \delta)\sharp\mathcal{M}(\delta, p')\sharp\mathcal{M}(p', p'') \\
&\overset{\textbf{(e)}}{=} \sum_{\delta', \delta} \sharp\mathcal{M}_N(\gamma, \delta')\sharp\mathcal{M}_N(\delta', \delta)\sharp\mathcal{M}(\delta, p'') + \sum_{\delta, p'} \sharp\mathcal{M}_N(\gamma, \delta)\sharp\mathcal{M}(\delta, p')\sharp\mathcal{M}(p', p'') \\
&= \sum_{\delta'} \sharp\mathcal{M}_N(\gamma, \delta') \left\{ \sum_{\delta} \sharp\mathcal{M}_N(\delta', \delta)\sharp\mathcal{M}(\delta, p'') + \sum_{p'} \sharp\mathcal{M}(\delta', p')\sharp\mathcal{M}(p', p'') \right\} \\
&\overset{\textbf{(c)}}{=} \sum_{\delta'} \sharp\mathcal{M}_N(\gamma, \delta')\sharp\partial\overline{\mathcal{M}}(\delta', p'') \\
&= 0.
\end{aligned}$$

Note that we use Theorem 4.1 **(e)** at $\overset{\textbf{(e)}}{=}$, and Theorem 4.1 **(c)** at $\overset{\textbf{(c)}}{=}$. Next we have

$$n(\gamma, \gamma'') = \sum_{\gamma'} \sharp\mathcal{M}_N(\gamma, \gamma')\sharp\mathcal{M}_N(\gamma', \gamma'') + \sum_{\delta, p'} \sharp\mathcal{M}_N(\gamma, \delta)\sharp\mathcal{M}(\delta, p')\sharp\mathcal{M}(p', \gamma'').$$

By Theorem 4.1 **(d)**, the first term is equal to $\sharp\partial\overline{\mathcal{M}}_N(\gamma, \gamma'')$, and the second term is equal to 0 since there is no broken negative trajectory from a negative boundary critical point δ to a positive boundary critical point γ'', and we obtain $n(\gamma, \gamma'') = 0$. Hence $\partial_{k-1} \circ \partial_k \gamma = 0$. $\square$

Note that we may also prove the invariance of Morse homology in terms of gradient trajectories, i.e., we may define a homotopy between Morse complexes, which induces an isomorphism of Morse homology. See the details in [1].

Next we prove the Leibniz rules in terms of gradient trees.

Let f_i be our Morse function on $M \setminus \partial M$, and $f_{i\partial M} : \partial M \to \mathbb{R}$ the boundary Morse function of f_i, for $i = 1, 2, 3$. We define the moduli spaces of gradient trees. Let p_1, p_2, p_3 be interior critical points of f_1, f_2, f_3, respectively. We denote by $\mathcal{M}(p_1, p_2, p_3)$ the set of gradient trees (l_1, l_2, l_3) such that

- $l_1 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_1/dt = -\overline{X}_{f_1}$ and $\lim_{t \to -\infty} l_1(t) = p_1$;
- $l_2 : (-\infty, 0] \to M \setminus \partial M$ satisfies $dl_2/dt = -\overline{X}_{f_2}$ and $\lim_{t \to -\infty} l_2(t) = p_2$;
- $l_3 : [0, \infty) \to M \setminus \partial M$ satisfies $dl_3/dt = -\overline{X}_{f_3}$ and $\lim_{t \to \infty} l_3(t) = p_3$; and
- $l_1(0) = l_2(0) = l_3(0)$.

Similarly, we define $\mathcal{M}(p_1, p_2, \gamma_3)$, $\mathcal{M}(p_1, \delta_2, p_3)$, $\mathcal{M}(\delta_1, p_2, p_3)$ and $\mathcal{M}(\delta_1, \delta_2, p_3)$. Note that $\mathcal{M}(p_1, \delta_2, \gamma_3)$, $\mathcal{M}(\delta_1, p_2, \gamma_3)$ and $\mathcal{M}(\delta_1, \delta_2, \gamma_3)$ are empty since there is no broken negative trajectory from a negative boundary critical point to a positive boundary critical point. Let γ_1, γ_2, γ_3 be positive boundary critical points of $f_{1\partial M}$, $f_{2\partial M}$, $f_{3\partial M}$, respectively. We denote by $\mathcal{M}_N(\gamma_1, \gamma_2, \gamma_3)$ the set of gradient trees (l_1, l_2, l_3) such that

- $l_1 : (-\infty, 0] \to \partial M$ satisfies $dl_1/dt = -\overline{X}_{f_1}$ and $\lim_{t \to -\infty} l_1(t) = \gamma_1$;
- $l_2 : (-\infty, 0] \to \partial M$ satisfies $dl_2/dt = -\overline{X}_{f_2}$ and $\lim_{t \to -\infty} l_1(t) = \gamma_2$;
- $l_3 : [0, \infty) \to \partial M$ satisfies $dl_3/dt = -\overline{X}_{f_3}$ and $\lim_{t \to -\infty} l_1(t) = \gamma_3$; and
- $l_1(0) = l_2(0) = l_3(0)$.

Similarly, we define $\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)$, $\mathcal{M}_N(\gamma_1,\delta_2,\delta_3)$, $\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)$ and $\mathcal{M}_N(\delta_1,\delta_2,\delta_3)$. Note that $\mathcal{M}_N(\gamma_1,\delta_2,\gamma_3)$, $\mathcal{M}_N(\delta_1,\gamma_2,\gamma_3)$ and $\mathcal{M}_N(\delta_1,\delta_2,\gamma_3)$ are empty since there is no broken negative gradient trajectories from a negative boundary critical point to a positive boundary critical point.

Note that we always use notation, for $i=1,2,3$,

- $p_i, p_i', p_i'' \in M \setminus \partial M$ for interior critical points of f_i;
- $\gamma_i, \gamma_i', \gamma_i'' \in \partial M$ for positive boundary critical points of $f_{i\partial M}$; and
- $\delta_i, \delta_i', \delta_i'' \in \partial M$ for negative boundary critical points of $f_{i\partial M}$.

Then we have the following theorem:

Theorem 4.3. *We may take generic f_i, for $i=1,2,3$, so that the following hold:*

(f) *$\mathcal{M}(p_1,p_2,p_3)$ is an orientable smooth manifold of dimension $\mu(p_1)+\mu(p_2)-\mu(p_3)-n$. If $\dim\mathcal{M}(p_1,p_2,p_3)=0$, then $\mathcal{M}(p_1,p_2,p_3)$ is compact. If $\mathcal{M}(p_1,p_2,p_3)=1$, then $\mathcal{M}(p_1,p_2,p_3)$ can be compactified into $\overline{\mathcal{M}}(p_1,p_2,p_3)$, whose boundary is*

$$
\begin{aligned}
\partial\overline{\mathcal{M}}(p_1,p_2,p_3) = & \bigcup_{\mu(p_1')=\mu(p_1)-1} \mathcal{M}(p_1,p_1')\times\mathcal{M}(p_1',p_2,p_3) \\
& \cup \bigcup_{\mu(p_2')=\mu(p_2)-1} \mathcal{M}(p_2,p_2')\times\mathcal{M}(p_1,p_2',p_3) \\
& \cup \bigcup_{\mu(p_3')=\mu(p_3)+1} \mathcal{M}(p_1,p_2,p_3')\times\mathcal{M}(p_3',p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_1)=\mu(p_1)-1\\ \mu(\delta_1)=\mu(\gamma_1)-1}} \mathcal{M}(p_1,\gamma_1)\times\mathcal{M}_N(\gamma_1,\delta_1)\times\mathcal{M}(\delta_1,p_2,p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_2)=\mu(p_2)-1\\ \mu(\delta_2)=\mu(\delta_2)-1}} \mathcal{M}(p_2,\gamma_2)\times\mathcal{M}_N(\gamma_2,\delta_2)\times\mathcal{M}(p_1,\delta_2,p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_3)=\mu(\delta_3)+1\\ \mu(\delta_3)=\mu(p_3)}} \mathcal{M}(p_1,p_2,\gamma_3)\times\mathcal{M}_N(\gamma_3,\delta_3)\times\mathcal{M}(\delta_3,p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_1)=\mu(p_1)-1\\ \mu(\gamma_2)=\mu(p_2)-1\\ \mu(\delta_3)=\mu(p_3)}} \mathcal{M}(p_1,\gamma_1)\times\mathcal{M}(p_2,\gamma_2)\times\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\times\mathcal{M}(\delta_3,p_3).
\end{aligned}
$$

(g) *$\mathcal{M}(p_1,p_2,\gamma_3)$ is an orientable smooth manifold of dimension $\mu(p_1)+\mu(p_2)-\mu(\gamma_3)-n$. If $\dim\mathcal{M}(p_1,p_2,\gamma_3)=0$, then $\mathcal{M}(p_1,p_2,\gamma_3)$ is compact. If $\mathcal{M}(p_1,p_2,\gamma_3)=$*

1, *then* $\mathcal{M}(p_1,p_2,\gamma_3)$ *can be compactified into* $\overline{\mathcal{M}}(p_1,p_2,\gamma_3)$, *whose boundary is*

$$\begin{aligned}
\partial\overline{\mathcal{M}}(p_1,p_2,\gamma_3) = & \bigcup_{\mu(p_1')=\mu(p_1)-1} \mathcal{M}(p_1,p_1')\times\mathcal{M}(p_1',p_2,\gamma_3) \\
& \cup \bigcup_{\mu(p_2')=\mu(p_2)-1} \mathcal{M}(p_2,p_2')\times\mathcal{M}(p_1,p_2',\gamma_3) \\
& \cup \bigcup_{\mu(p_3)=\mu(\gamma_3)+1} \mathcal{M}(p_1,p_2,p_3)\times\mathcal{M}(p_3,\gamma_3) \\
& \cup \bigcup_{\mu(\gamma_3')=\mu(\gamma_3)+1} \mathcal{M}(p_1,p_2,\gamma_3')\times\mathcal{M}_N(\gamma_3',\gamma_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_1)=\mu(p_1)-1\\ \mu(\gamma_2)=\mu(p_2)-1}} \mathcal{M}(p_1,\gamma_1)\times\mathcal{M}(p_2,\gamma_2)\times\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3).
\end{aligned}$$

(h) $\mathcal{M}(p_1,\delta_2,p_3)$ *is an orientable smooth manifold of dimension* $\mu(p_1)+\mu(\delta_2)-\mu(p_3)-n+1$. *If* $\dim\mathcal{M}(p_1,\delta_2,p_3)=0$, *then* $\mathcal{M}(p_1,\delta_2,p_3)$ *is compact. If* $\mathcal{M}(p_1,\delta_2,p_3)=1$, *then* $\mathcal{M}(p_1,\delta_2,p_3)$ *can be compactified into* $\overline{\mathcal{M}}(p_1,\delta_2,p_3)$, *whose boundary is*

$$\begin{aligned}
\partial\overline{\mathcal{M}}(p_1,\delta_2,p_3) = & \bigcup_{\mu(p_1')=\mu(p_1)-1} \mathcal{M}(p_1,p_1')\times\mathcal{M}(p_1',\delta_2,p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_1)=\mu(p_1)-1\\ \mu(\delta_1)=\mu(\gamma_1)-1}} \mathcal{M}(p_1,\gamma_1)\times\mathcal{M}_N(\gamma_1,\delta_1)\times\mathcal{M}(\delta_1,\delta_2,p_3) \\
& \cup \bigcup_{\mu(p_2)=\mu(\delta_2)} \mathcal{M}(\delta_2,p_2)\times\mathcal{M}(p_1,p_2,p_3) \\
& \cup \bigcup_{\mu(\delta_2')=\mu(\delta_2)-1} \mathcal{M}_N(\delta_2,\delta_2')\times\mathcal{M}(p_1,\delta_2',p_3) \\
& \cup \bigcup_{\mu(p_3')=\mu(p_3)+1} \mathcal{M}(p_1,\delta_2,p_3')\times\mathcal{M}(p_3',p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_1)=\mu(p_1)-1\\ \mu(\delta_3)=\mu(p_3)}} \mathcal{M}(p_1,\gamma_1)\times\mathcal{M}_N(\gamma_1,\delta_2,\delta_3)\times\mathcal{M}(\delta_3,p_3).
\end{aligned}$$

(i) $\mathcal{M}(\delta_1,p_2,p_3)$ *is an orientable smooth manifold of dimension* $\mu(\delta_1)+\mu(p_2)-\mu(p_3)-n+1$. *If* $\dim\mathcal{M}(\delta_1,p_2,p_3)=0$, *then* $\mathcal{M}(\delta_1,p_2,p_3)$ *is compact. If* $\mathcal{M}(\delta_1,p_2,p_3)=$

1, then $\mathcal{M}(\delta_1, p_2, p_3)$ can be compactified into $\overline{\mathcal{M}}(\delta_1, p_2, p_3)$, whose boundary is

$$\begin{aligned}
\partial\overline{\mathcal{M}}(\delta_1, p_2, p_3) = & \bigcup_{\mu(p_1)=\mu(\delta_1)} \mathcal{M}(\delta_1, p_1) \times \mathcal{M}(p_1, p_2, p_3) \\
& \cup \bigcup_{\mu(\delta_1')=\mu(\delta_1)-1} \mathcal{M}_N(\delta_1, \delta_1') \times \mathcal{M}(\delta_1', p_2, p_3) \\
& \cup \bigcup_{\mu(p_2')=\mu(p_2)-1} \mathcal{M}(p_2, p_2') \times \mathcal{M}(\delta_1, p_2', p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_2)=\mu(p_2)-1 \\ \mu(\delta_2)=\mu(\gamma_2)-1}} \mathcal{M}(p_2, \gamma_2) \times \mathcal{M}_N(\gamma_2, \delta_2) \times \mathcal{M}(\delta_1, \delta_2, p_3) \\
& \cup \bigcup_{\mu(p_3')=\mu(p_3)+1} \mathcal{M}(\delta_1, p_2, p_3') \times \mathcal{M}(p_3', p_3) \\
& \cup \bigcup_{\substack{\mu(\gamma_2)=\mu(p_2)-1 \\ \mu(\delta_3)=\mu(p_3)}} \mathcal{M}(p_2, \gamma_2) \times \mathcal{M}_N(\delta_1, \gamma_2, \delta_3) \times \mathcal{M}(\delta_3, p_3).
\end{aligned}$$

(j) *$\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)$ is an orientable smooth manifold of dimension $\mu(\gamma_1)+\mu(\gamma_2)-\mu(\delta_3)-n+1$. If $\dim \mathcal{M}_N(\gamma_1, \gamma_2, \delta_3) = 0$, then $\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)$ is compact. If $\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3) = 1$, then $\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)$ can be compactified into $\overline{\mathcal{M}}_N(\gamma_1, \gamma_2, \delta_3)$, whose boundary is*

$$\begin{aligned}
\partial\overline{\mathcal{M}}_N(\gamma_1, \gamma_2, \delta_3) = & \bigcup_{\mu(\gamma_1')=\mu(\gamma_1)-1} \mathcal{M}_N(\gamma_1, \gamma_1') \times \mathcal{M}_N(\gamma_1', \gamma_2, \delta_3) \\
& \cup \bigcup_{\mu(\delta_1)=\mu(\gamma_1)-1} \mathcal{M}_N(\gamma_1, \delta_1) \times \mathcal{M}_N(\delta_1, \gamma_2, \delta_3) \\
& \cup \bigcup_{\mu(\gamma_2')=\mu(\gamma_2)-1} \mathcal{M}_N(\gamma_2, \gamma_2') \times \mathcal{M}_N(\gamma_1, \gamma_2', \delta_3) \\
& \cup \bigcup_{\mu(\delta_2)=\mu(\gamma_2)-1} \mathcal{M}_N(\gamma_2, \delta_2) \times \mathcal{M}_N(\gamma_1, \delta_2, \delta_3) \\
& \cup \bigcup_{\mu(\gamma_3)=\mu(\delta_3)+1} \mathcal{M}_N(\gamma_1, \gamma_2, \gamma_3) \times \mathcal{M}_N(\gamma_3, \delta_3) \\
& \cup \bigcup_{\mu(\delta_3')=\mu(\delta_3)+1} \mathcal{M}_N(\gamma_1, \gamma_2, \delta_3') \times \mathcal{M}_N(\delta_3', \delta_3).
\end{aligned}$$

(k) *$\mathcal{M}(\delta_1, \delta_2, p_3)$ is an orientable smooth manifold of dimension $\mu(\delta_1)+\mu(\delta_2)-\mu(p_3)-n+2$. If $\dim \mathcal{M}(\delta_1, \delta_2, p_3) = 0$, then $\mathcal{M}(\delta_1, \delta_2, p_3)$ is compact. If $\mathcal{M}(\delta_1, \delta_2, p_3) =$*

1, then $\mathcal{M}(\delta_1,\delta_2,p_3)$ *can be compactified into* $\overline{\mathcal{M}}(\delta_1,\delta_2,p_3)$*, whose boundary is*

$$\begin{aligned}\partial\overline{\mathcal{M}}(\delta_1,\delta_2,p_3) = &\bigcup_{\mu(\delta_1')=\mu(\delta_1)-1} \mathcal{M}_N(\delta_1,\delta_1')\times\mathcal{M}(\delta_1',\delta_2,p_3)\\ &\cup \bigcup_{\mu(p_1)=\mu(\delta_1)} \mathcal{M}(\delta_1,p_1)\times\mathcal{M}(p_1,\delta_2,p_3)\\ &\cup \bigcup_{\mu(\delta_2')=\mu(\delta_2)-1} \mathcal{M}_N(\delta_2,\delta_2')\times\mathcal{M}(\delta_1,\delta_2',p_3)\\ &\cup \bigcup_{\mu(p_2)=\mu(\delta_2)} \mathcal{M}(\delta_2,p_2)\times\mathcal{M}(\delta_1,p_2,p_3)\\ &\cup \bigcup_{\mu(p_3')=\mu(p_3)+1} \mathcal{M}(\delta_1,\delta_2,p_3')\times\mathcal{M}(p_3',p_3)\\ &\cup \bigcup_{\mu(\delta_3)=\mu(p_3)} \mathcal{M}_N(\delta_1,\delta_2,\delta_3)\times\mathcal{M}(\delta_3,p_3).\end{aligned}$$

(l) $\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)$ *is an orientable smooth manifold of dimension* $\mu(\delta_1)+\mu(\gamma_2)-\mu(\delta_3)-n+1$*. If* $\dim\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)=0$*, then* $\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)$ *is compact. If* $\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)=1$*, then* $\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)$ *can be compactified into* $\overline{\mathcal{M}}_N(\delta_1,\gamma_2,\delta_3)$*, whose boundary is*

$$\begin{aligned}\partial\overline{\mathcal{M}}_N(\delta_1,\gamma_2,\delta_3) = &\bigcup_{\mu(\delta_1')=\mu(\delta_1)-1} \mathcal{M}_N(\delta_1,\delta_1')\times\mathcal{M}_N(\delta_1',\gamma_2,\delta_3)\\ &\cup \bigcup_{\mu(\gamma_2')=\mu(\gamma_2)-1} \mathcal{M}_N(\gamma_2,\gamma_2')\times\mathcal{M}_N(\delta_1,\gamma_2',\delta_3)\\ &\cup \bigcup_{\mu(\delta_2)=\mu(\gamma_2)-1} \mathcal{M}_N(\gamma_2,\delta_2)\times\mathcal{M}_N(\delta_1,\delta_2,\delta_3)\\ &\cup \bigcup_{\mu(\delta_3')=\mu(\delta_3)+1} \mathcal{M}_N(\delta_2,\gamma_2,\delta_3')\times\mathcal{M}_N(\delta_3',\delta_3).\end{aligned}$$

(m) $\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)$ *is an orientable smooth manifold of dimension* $\mu(\gamma_1)+\mu(\gamma_2)-\mu(\gamma_3)-n+1$*. If* $\dim\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)=0$*, then* $\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)$ *is compact. If* $\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)=1$*, then* $\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)$ *can be compactified into* $\overline{\mathcal{M}}_N(\gamma_1,\gamma_2,\gamma_3)$*, whose boundary is*

$$\begin{aligned}\partial\overline{\mathcal{M}}_N(\gamma_1,\gamma_2,\gamma_3) = &\bigcup_{\mu(\gamma_1')=\mu(\gamma_1)-1} \mathcal{M}_N(\gamma_1,\gamma_1')\times\mathcal{M}_N(\gamma_1',\gamma_2,\gamma_3)\\ &\cup \bigcup_{\mu(\gamma_2')=\mu(\gamma_2)-1} \mathcal{M}_N(\gamma_2,\gamma_2')\times\mathcal{M}_N(\gamma_1,\gamma_2',\gamma_3)\\ &\cup \bigcup_{\mu(\gamma_3')=\mu(\gamma)+1} \mathcal{M}_N(\gamma_2,\gamma_2,\gamma_3')\times\mathcal{M}_N(\gamma_3',\gamma_3).\end{aligned}$$

Note that we put the orientation on moduli spaces which comes from the intersection number of $U^*_*, I^*_* : e^*_* \to M$ *and* S^*_**.*

We omit the proof of Theorem 4.3. We may list every boundary components of 1-dimensional moduli spaces in Theorem 4.3 without omission by chasing critical points so that we obtain 1-dimensional moduli spaces after gluing gradient trees.

Note that there is no broken negative trajectory from a negative boundary critical point to a positive boundary critical point.

Remember that we defined the linear map $m_2 : C_{k_1}(f_1) \otimes C_{k_2}(f_2) \to C_{k_1+k_2-n}(f_3)$ by

$$
\begin{aligned}
m_2(p_1, p_2) &:= \sum_{p_3} \sharp\mathcal{M}(p_1, p_2, p_3)p_3 + \sum_{\gamma_3} \sharp\mathcal{M}(p_1, p_2, \gamma_3)\gamma_3, \\
m_2(p_1, \gamma_2) &:= \sum_{\gamma_1, \delta_3, p_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3)p_3 \\
&\quad + \sum_{\delta_2, p_3} \sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}(p_1, \delta_2, p_3)p_3 \\
&\quad + \sum_{\gamma_1, \gamma_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \gamma_3)\gamma_3, \\
m_2(\gamma_1, p_2) &:= \sum_{\delta_1, p_3} \sharp\mathcal{M}_N(\gamma_1, \delta_1)\sharp\mathcal{M}(\delta_1, p_2, p_3)p_3, \\
m_2(\gamma_1, \gamma_2) &:= \sum_{\delta_1, \delta_3, p_3} \sharp\mathcal{M}_N(\gamma_1, \delta_1)\sharp\mathcal{M}_N(\delta_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3)p_3 \\
&\quad + \sum_{\delta_1, \delta_2, p_3} \sharp\mathcal{M}_N(\gamma_1, \delta_1)\sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}(\delta_1, \delta_2, p_3)p_3.
\end{aligned}
$$

We already proved the Leibniz rules in Theorem 3.3 by considering intersection of $U^*_*, I^*_* : e^*_* \to M, S^*_*$ and their images by ψ_ε. Here we prove the Leibniz rules by observing the boundary of 1-dimensional moduli spaces of gradient trees.

Theorem 4.4. *We denote by $\partial^{f_1}, \partial^{f_2}$ and ∂^{f_3} the boundary operators of Morse complex for f_1, f_2 and f_3, respectively. For interior critical points p_1, p_2 of f_1, f_2, respectively, we obtain the Leibniz rule: (We omit the sign convention.)*

$$\partial^{f_3} m_2(p_1, p_2) = m_2(\partial^{f_1} p_1, p_2) \pm m_2(p_1, \partial^{f_2} p_2).$$

Proof. First we calculate $\partial^{f_3} m_2(p_1, p_2)$.

$$
\begin{aligned}
\partial^{f_3} m_2(p_1, p_2) &= \partial^{f_3}\left\{\sum_{p_3} \sharp\mathcal{M}(p_1, p_2, p_3)p_3 + \sum_{\gamma_3} \sharp\mathcal{M}(p_1, p_2, \gamma_3)\gamma_3\right\} \\
&= \sum_{p_3, p_3'} \sharp\mathcal{M}(p_1, p_2, p_3)\sharp\mathcal{M}(p_3, p_3')p_3' \\
&\quad + \sum_{p_3, \gamma_3'} \sharp\mathcal{M}(p_1, p_2, p_3)\sharp\mathcal{M}(p_3, \gamma_3')\gamma_3' \\
&\quad + \sum_{\gamma_3, \gamma_3'} \sharp\mathcal{M}(p_1, p_2, \gamma_3)\sharp\mathcal{M}_N(\gamma_3, \gamma_3')\gamma_3' \\
&\quad + \sum_{\gamma_3, \delta_3, p_3'} \sharp\mathcal{M}(p_1, p_2, \gamma_3)\sharp\mathcal{M}_N(\gamma_3, \delta_3)\sharp\mathcal{M}(\delta_3, p_3')p_3'.
\end{aligned}
$$

Next we calculate $m_2(\partial^{f_1}p_1, p_2)$ and $m_2(p_1, \partial^{f_2}p_2)$.

$$\begin{aligned} m_2(\partial^{f_1}p_1, p_2) = m_2 &\left(\sum_{p_1'} \sharp\mathcal{M}(p_1, p_1')p_1' + \sum_{\gamma_1} \sharp\mathcal{M}(p_1, \gamma_1)\gamma_1, p_2 \right) \\ = &\sum_{p_1', p_3'} \sharp\mathcal{M}(p_1, p_1')\sharp\mathcal{M}(p_1', p_2, p_3')p_3' \\ &+ \sum_{p_1', \gamma_3'} \sharp\mathcal{M}(p_1, p_1')\sharp\mathcal{M}(p_1', p_2, \gamma_3')\gamma_3' \\ &+ \sum_{\gamma_1, \delta_1, p_3'} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \delta_1)\sharp\mathcal{M}(\delta_1, p_2, p_3')p_3'. \end{aligned}$$

$$\begin{aligned} m_2(p_1, \partial^{f_2}p_2) = m_2 &\left(p_1, \sum_{p_2'} \sharp\mathcal{M}(p_2, p_2')p_2' + \sum_{\gamma_2} \sharp\mathcal{M}(p_2, \gamma_2)\gamma_2 \right) \\ = &\sum_{p_2', p_3'} \sharp\mathcal{M}(p_2, p_2')\sharp\mathcal{M}(p_1, p_2', p_3')p_3' \\ &+ \sum_{p_2', \gamma_3'} \sharp\mathcal{M}(p_2, p_2')\sharp\mathcal{M}(p_1, p_2', \gamma_3')\gamma_3' \\ &+ \sum_{\gamma_2, p_3', \gamma_1, \delta_3} \sharp\mathcal{M}(p_2, \gamma_2)\sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3')p_3' \\ &+ \sum_{\gamma_2, p_3', \delta_2} \sharp\mathcal{M}(p_2, \gamma_2)\sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}(p_1, \delta_2, p_3')p_3' \\ &+ \sum_{\gamma_2, \gamma_3', \gamma_1} \sharp\mathcal{M}(p_2, \gamma_2)\sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \gamma_3')\gamma_3'. \end{aligned}$$

We define $n(p_1, p_2, p_3')$ and $n(p_1, p_2, \gamma_3')$ by

$$\begin{aligned} \partial^{f_3}m_2(p_1, p_2) - &m_2(\partial^{f_1}p_1, p_2) \pm m_2(p_1, \partial^{f_2}p_2) \\ &= \sum_{p_3'} n(p_1, p_2, p_3')p_3' + \sum_{\gamma_3'} n(p_1, p_2, \gamma_3')\gamma_3'. \end{aligned}$$

Then

$$
\begin{aligned}
&n(p_1,p_2,p_3') \\
&\quad= \sum_{p_3} \sharp\mathcal{M}(p_1,p_2,p_3)\sharp\mathcal{M}(p_3,p_3') + \sum_{\gamma_3,\delta_3} \sharp\mathcal{M}(p_1,p_2,\gamma_3)\sharp\mathcal{M}_N(\gamma_3,\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \\
&\qquad+ \sum_{p_1'} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}(p_1',p_2,p_3') + \sum_{\gamma_1,\delta_1} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3') \\
&\qquad+ \sum_{p_2'} \sharp\mathcal{M}(p_2,p_2')\sharp\mathcal{M}(p_1,p_2',p_3') + \sum_{\gamma_2,\delta_2} \sharp\mathcal{M}(p_2,\gamma_2)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p_3') \\
&\qquad+ \sum_{\gamma_1,\gamma_2,\delta_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}(p_2,\gamma_2)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \\
&\quad\overset{\textbf{(f)}}{=} \sharp\partial\overline{\mathcal{M}}(p_1,p_2,p_3) \\
&\quad= 0.
\end{aligned}
$$

Note that we use Theorem 4.3 **(f)** at $\overset{\textbf{(f)}}{=}$. Moreover,

$$
\begin{aligned}
n(p_1,p_2,\gamma_3') &= \sum_{p_3} \sharp\mathcal{M}(p_1,p_2,p_3)\sharp\mathcal{M}(p_3,\gamma_3') + \sum_{\gamma_3} \sharp\mathcal{M}(p_1,p_2,\gamma_3)\sharp\mathcal{M}_N(\gamma_3,\gamma_3') \\
&\quad+ \sum_{p_1'} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}(p_1',p_2,\gamma_3') + \sum_{p_2'} \sharp\mathcal{M}(p_2,p_2')\sharp\mathcal{M}(p_1,p_2',\gamma_3') \\
&\quad+ \sum_{\gamma_1,\gamma_2} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}(p_2,\gamma_2)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3') \\
&\overset{\textbf{(g)}}{=} \sharp\partial\overline{\mathcal{M}}(p_1,p_2,\gamma_3') \\
&= 0.
\end{aligned}
$$

Note that we use Theorem 4.3 **(g)** at $\overset{\textbf{(g)}}{=}$. Hence we obtain $\partial^{f_3}m_2(p_1,p_2) = m_2(\partial^{f_1}p_1,p_2) \pm m_2(p_1,\partial^{f_2}p_2)$. □

Theorem 4.5. *For an interior critical point p_1 of f_1 and a positive boundary critical point γ_2 of $f_{2\partial M}$, we obtain the Leibniz rule: (We omit the sign convention.)*

$$\partial^{f_3}m_2(p_1,\gamma_2) = m_2(\partial^{f_1}p_1,\gamma_2) \pm m_2(p_1,\partial^{f_2}\gamma_2).$$

Proof. First we calculate $\partial^{f_3} m_2(p_1,\gamma_2)$.

$$
\begin{aligned}
\partial^{f_3} m_2(p_1,\gamma_2) = \partial^{f_1} \Bigg\{ & \sum_{\gamma_1,\delta_3,p_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)p_3 \\
& + \sum_{\delta_2,p_3} \sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p_3)p_3 \\
& + \sum_{\gamma_1,\gamma_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)\gamma_3 \Bigg\} \\
= & \sum_{p_3',\gamma_1,\delta_3,p_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)\sharp\mathcal{M}(p_3,p_3')p_3' \\
& + \sum_{\gamma_3',\gamma_1,\delta_3,p_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)\sharp\mathcal{M}(p_3,\gamma_3')\gamma_3' \qquad \text{(A)} \\
& + \sum_{p_3',\delta_2,p_3} \sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p_3)\sharp\mathcal{M}(p_3,p_3')p_3' \\
& + \sum_{\gamma_3',\delta_2,p_3} \sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p_3)\sharp\mathcal{M}(p_3,\gamma_3')\gamma_3' \qquad \text{(B)} \\
& + \sum_{\gamma_3',\gamma_1,\gamma_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)\sharp\mathcal{M}_N(\gamma_3,\gamma_3')\gamma_3' \\
& + \sum_{p_3',\gamma_1,\gamma_3,\delta_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)\sharp\mathcal{M}_N(\gamma_3,\delta_3)\sharp\mathcal{M}(\delta_3,p_3')p_3'.
\end{aligned}
$$

Note that the line (A) is equal to 0 since there is no broken negative trajectory from a negative boundary critical point δ_3 to a positive boundary critical point γ_3', and similarly, the line (B) is equal to 0 since there is no broken negative trajectory from a negative boundary critical point δ_2 to a positive boundary critical point γ_3'.

Next we calculate $m_2(\partial^{f_1} p_1,\gamma_2)$ and $m_2(p_1,\partial^{f_2}\gamma_2)$.

$$
\begin{aligned}
m_2(\partial^{f_1} p_1,\gamma_2) = & \, m_2\left(\sum_{p_1'} \sharp\mathcal{M}(p_1,p_1')p_1' + \sum_{\gamma_1} \sharp\mathcal{M}(p_1,\gamma_1)\gamma_1, \gamma_2 \right) \\
= & \sum_{p_1',p_3',\gamma_1,\delta_3} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}(p_1',\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3')p_3' \\
& + \sum_{p_1',p_3',\delta_2} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1',\delta_2,p_3')p_3' \\
& + \sum_{p_1',\gamma_3',\gamma_1} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}(p_1',\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3')\gamma_3' \\
& + \sum_{\gamma_1,p_3',\delta_1,\delta_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3')p_3' \\
& + \sum_{\gamma_1,\delta_1,\delta_2} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3')p_3'.
\end{aligned}
$$

$$\begin{aligned}
m_2(p_1, \partial^{f_2}\gamma_2) &= m_2\left(p_1, \sum_{\gamma_2'} \sharp\mathcal{M}_N(\gamma_2, \gamma_2')\gamma_2' + \sum_{\delta_2, p_2} \sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}(\delta_2, p_2)p_2\right) \\
&= \sum_{\gamma_2', p_3', \gamma_1, \delta_3} \sharp\mathcal{M}_N(\gamma_2, \gamma_2')\sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2', \delta_3)\sharp\mathcal{M}(\delta_3, p_3')p_3' \\
&\quad + \sum_{\gamma_2', \delta_2} \sharp\mathcal{M}_N(\gamma_2, \gamma_2')\sharp\mathcal{M}_N(\gamma_2', \delta_2)\sharp\mathcal{M}(p_1, \delta_2, p_3')p_3' \\
&\quad + \sum_{\gamma_2', \gamma_3', \gamma_1} \sharp\mathcal{M}_N(\gamma_2, \gamma_2')\sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2', \gamma_3')\gamma_3' \\
&\quad + \sum_{\delta_2, p_2, p_3'} \sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}(\delta_2, p_2)\sharp\mathcal{M}(p_1, p_2, p_3')p_3' \\
&\quad + \sum_{\delta_2, p_2, \gamma_3} \sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}(\delta_2, p_2)\sharp\mathcal{M}(p_1, p_2, \gamma_3')\gamma_3'. \qquad \text{(C)}
\end{aligned}$$

Note that the line (C) is equal to 0 since there is no broken negative gradient trajectory from a negative boundary critical point δ_2 to a positive boundary critical point γ_3'.

We define $n(p_1, \gamma_2, p_3')$ and $n(p_1, \gamma_2, \gamma_3')$ by

$$\begin{aligned}
\partial^{f_3} m_2(p_1, \gamma_2) - m_2(\partial^{f_1} p_2, \gamma_2) \pm m_2(p_1, \partial^{f_2}\gamma_2) \\
= \sum_{p_3'} n(p_1, \gamma_2, p_3')p_3' + \sum_{\gamma_3'} n(p_1, \gamma_2, \gamma_3')\gamma_3'.
\end{aligned}$$

Then

$$
\begin{aligned}
n(p_1,\gamma_2,p_3') = &\sum_{\gamma_1,\delta_3,p_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)\sharp\mathcal{M}(p_3,p_3') \\
&+\sum_{\delta_2,p_3} \sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p_3)\sharp\mathcal{M}(p_3,p_3') \qquad \text{(D)} \\
&+\sum_{\gamma_1,\gamma_3,\delta_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3)\sharp\mathcal{M}_N(\gamma_3,\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \\
&+\sum_{p_1',\gamma_1,\delta_3} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}(p_1',\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \\
&+\sum_{p_1',\delta_2} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1',\delta_2,p_3') \qquad \text{(E)} \\
&+\sum_{\delta_1,\gamma_3,\delta_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \\
&+\sum_{\delta_1,\delta_2,\gamma_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3') \qquad \text{(F)} \\
&+\sum_{\gamma_1,\gamma_2',\delta_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_2,\gamma_2')\sharp\mathcal{M}_N(\gamma_1,\gamma_2',\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \\
&+\sum_{\gamma_2',\delta_2} \sharp\mathcal{M}_N(\gamma_2,\gamma_2')\sharp\mathcal{M}_N(\gamma_2',\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p_3') \qquad \text{(G)} \\
&+\sum_{\delta_2,p_2} \sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_2,p_2)\sharp\mathcal{M}(p_1,p_2,p_3'). \qquad \text{(H)}
\end{aligned}
$$

By Theorem 4.1 **(e)**, the line (G) is equal to

$$
\sum_{\gamma_2',\delta_2} \left\{ \sharp\partial\overline{\mathcal{M}}_N(\gamma_2,\delta_2) + \sum_{\delta_2'} \sharp\mathcal{M}_N(\gamma_2,\delta_2')\sharp\mathcal{M}_N(\delta_2',\delta_2) \right\} \sharp\mathcal{M}(p_1,\delta_2,p_3'). \qquad \text{(I)}
$$

Note that $\sharp\partial\overline{\mathcal{M}}(p_1,\delta_2)$ is equal to 0. Then, by Theorem 4.3 **(h)**, the sum of the lines (D), (E), (F), (H) and (I) is equal to

$$
\sum_{\delta_2} \sharp\mathcal{M}_N(\gamma_2,\delta_2) \left\{ \sharp\partial\overline{\mathcal{M}}(p_1,\delta_2,p_3') + \sum_{\gamma_1,\delta_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\delta_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \right\}. \qquad \text{(J)}
$$

Note that $\sharp\partial\overline{\mathcal{M}}(p_1, \delta_2, p_3')$ is equal to 0. Hence $n(p_1, \gamma_2, p_3')$ is equal to

$$\begin{aligned}
n(p_1, \gamma_2, p_3') = &\sum_{\gamma_1, \delta_3, p_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3)\sharp\mathcal{M}(p_3, p_3') \qquad \text{(K)}\\
&+ \sum_{\gamma_1, \gamma_3, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \gamma_3)\sharp\mathcal{M}_N(\gamma_3, \delta_3)\sharp\mathcal{M}(\delta_3, p_3')\\
&+ \sum_{p_1', \gamma_1, \delta_3} \sharp\mathcal{M}(p_1, p_1')\sharp\mathcal{M}(p_1', \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3') \qquad \text{(L)}\\
&+ \sum_{\delta_1, \gamma_3, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \delta_1)\sharp\mathcal{M}_N(\delta_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3')\\
&+ \sum_{\gamma_1, \gamma_2', \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_2, \gamma_2')\sharp\mathcal{M}_N(\gamma_1, \gamma_2', \delta_3)\sharp\mathcal{M}(\delta_3, p_3')\\
&+ \sum_{\gamma_1, \delta_2, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}_N(\gamma_1, \delta_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3').
\end{aligned}$$

Moreover, by Theorem 4. 1 **(c)**, the line (K) is equal to

$$\sum_{\gamma_1, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3) \left\{ \sharp\partial\overline{\mathcal{M}}(\delta_3, p_3') + \sum_{\delta_3'} \sharp\mathcal{M}_N(\delta_3, \delta_3')\sharp\mathcal{M}(\delta_3', p_3') \right\}, \tag{M}$$

and, by Theorem 4.1 **(b)**, the line (L) is equal to

$$\sum_{\gamma_1, \delta_3} \left\{ \sharp\partial\overline{\mathcal{M}}(p_1, \gamma_1) + \sum_{\gamma_1'} \sharp\mathcal{M}(p_1, \gamma_1')\sharp\mathcal{M}_N(\gamma_1', \gamma_1) \right\} \sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3'). \tag{N}$$

Note that $\sharp\partial\overline{\mathcal{M}}(\delta_3, p_3')$ and $\sharp\partial\overline{\mathcal{M}}(p_1, \gamma_1)$ are equal to 0. Hence $n(p_1, \gamma_2, p_3')$ is equal to

$$\begin{aligned}
n(p_1, \gamma_2, p_3') = &\sum_{\gamma_1, \delta_3, \delta_3'} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}_N(\delta_3, \delta_3')\sharp\mathcal{M}(\delta_3', p_3')\\
&+ \sum_{\gamma_1, \gamma_3, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \gamma_3)\sharp\mathcal{M}_N(\gamma_3, \delta_3)\sharp\mathcal{M}(\delta_3, p_3')\\
&+ \sum_{\gamma_1', \gamma_1, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1')\sharp\mathcal{M}_N(\gamma_1', \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3')\\
&+ \sum_{\delta_1, \gamma_3, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_1, \delta_1)\sharp\mathcal{M}_N(\delta_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3')\\
&+ \sum_{\gamma_1, \gamma_2', \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_2, \gamma_2')\sharp\mathcal{M}_N(\gamma_1, \gamma_2', \delta_3)\sharp\mathcal{M}(\delta_3, p_3')\\
&+ \sum_{\gamma_1, \delta_2, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\mathcal{M}_N(\gamma_2, \delta_2)\sharp\mathcal{M}_N(\gamma_1, \delta_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3').
\end{aligned}$$

Then, by Theorem 4.3 **(j)**, $n(p_1, \gamma_2, p_3')$ is equal to

$$\sum_{\gamma_1, \delta_3} \sharp\mathcal{M}(p_1, \gamma_1)\sharp\partial\overline{\mathcal{M}}_N(\gamma_1, \gamma_2, \delta_3)\sharp\mathcal{M}(\delta_3, p_3') = 0.$$

Next we have

$$\begin{aligned}
n(p_1,\gamma_2,\gamma_3') &= \sum_{\gamma_1,\gamma_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3')\sharp\mathcal{M}_N(\gamma_3,\gamma_3') \\
&\quad + \sum_{p_1',\gamma_1} \sharp\mathcal{M}(p_1,p_1')\sharp\mathcal{M}(p_1',\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3') \\
&\quad + \sum_{\gamma_1,\gamma_2'} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_2,\gamma_2')\sharp\mathcal{M}_N(\gamma_1,\gamma_2',\gamma_3') \\
&\overset{(\mathbf{b})}{=} \sum_{\gamma_1,\gamma_3} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3')\sharp\mathcal{M}_N(\gamma_3,\gamma_3') \\
&\quad + \sum_{\gamma_1',\gamma_1} \sharp\mathcal{M}(p_1,\gamma_1')\sharp\mathcal{M}_N(\gamma_1',\gamma_1)\sharp\mathcal{M}_N(\gamma_1,\gamma_2,\gamma_3') \\
&\quad + \sum_{\gamma_1,\gamma_2'} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\mathcal{M}_N(\gamma_2,\gamma_2')\sharp\mathcal{M}_N(\gamma_1,\gamma_2',\gamma_3') \\
&\overset{(\mathbf{m})}{=} \sum_{\gamma_1} \sharp\mathcal{M}(p_1,\gamma_1)\sharp\partial\overline{\mathcal{M}}_N(\gamma_1,\gamma_2,\gamma_3') \\
&= 0.
\end{aligned}$$

Note that we use Theorem 4.1 **(b)** at $\overset{(\mathbf{b})}{=}$ and Theorem 4.3 **(m)** at $\overset{(\mathbf{m})}{=}$. Hence we obtain $\partial^{f_3}m_2(p_1,\gamma_2) = m_2(\partial^{f_1}p_1,\gamma_2) \pm m_2(p_1,\partial^{f_2}\gamma_2)$. □

Theorem 4.6. *For a positive boundary critical point γ_1 of $f_{1\partial M}$ and an interior critical point p_2 of f_2, we obtain the Leibniz rule: (We omit the sign convention.)*

$$\partial^{f_3}m_2(\gamma_1,p_2) = m_2(\partial^{f_1}\gamma_1,p_2) \pm m_2(\gamma_1,\partial^{f_2}p_2).$$

Proof. First we calculate $\partial^{f_3}m_2(\gamma_1,p_2)$.

$$\begin{aligned}
\partial^{f_3}m_2(\gamma_1,p_2) &= \partial^{f_3}\left\{\sum_{p_3,\delta_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3)p_3\right\} \\
&= \sum_{p_3',\delta_1,p_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3)\sharp\mathcal{M}(p_3,p_3')p_3' \\
&\quad + \sum_{\gamma_3,\delta_1,p_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3)\sharp\mathcal{M}(p_3,\gamma_3)\gamma_3. \qquad \text{(O)}
\end{aligned}$$

Note that the line (O) is equal to 0 since there is no broken negative gradient trajectory from a negative boundary critical point δ_1 to a positive boundary critical point γ_3.

Next we calculate $m_2(\partial^{f_1}\gamma_1, p_2)$ and $m_2(\gamma_1, \partial^{f_2} p_2)$.

$$
\begin{aligned}
m_2(\partial^{f_1}\gamma_1, p_2) &= m_2\left(\sum_{\gamma_1'} \sharp\mathcal{M}_N(\gamma_1,\gamma_1')\gamma_1' + \sum_{\delta_1,p_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)p_1, p_2\right)\\
&= \sum_{p_3',\gamma_1',\delta_1} \sharp\mathcal{M}_N(\gamma_1,\gamma_1')\sharp\mathcal{M}_N(\gamma_1',\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3')p_3' \qquad \text{(P)}\\
&\quad + \sum_{p_3',\delta_1,p_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)\sharp\mathcal{M}(p_1,p_2,p_3')p_3'\\
&\quad + \sum_{\gamma_3,\delta_1,p_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)\sharp\mathcal{M}(p_1,p_2,\gamma_3)\gamma_3. \qquad \text{(Q)}
\end{aligned}
$$

Note that, by Theorem 4.1 **(e)**, the line (P) is equal to

$$
\sum_{p_3',\delta_1',\delta_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1')\sharp\mathcal{M}_N(\delta_1',\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3')p_3'.
$$

Moreover, the line (Q) is equal to 0 since there is no broken negative gradient trajectory from a negative boundary critical point δ_1 to a positive boundary critical point γ_3. Moreover,

$$
\begin{aligned}
m_2(\gamma_1, \partial^{f_2} p_2) &= m_2\left(\gamma_1, \sum_{p_2'} \sharp\mathcal{M}(p_2,p_2')p_2' + \sum_{\gamma_2} \sharp\mathcal{M}(p_2,\gamma_2)\gamma_2\right)\\
&= \sum_{p_3',\delta_1,p_2'} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(p_2,p_2')\sharp\mathcal{M}(\delta_1,p_2',p_3')p_3'\\
&\quad + \sum_{p_3',\delta_1,\gamma_2,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(p_2,\gamma_2)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3')p_3'\\
&\quad + \sum_{p_3',\delta_1,\gamma_2,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(p_2,\gamma_2)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3')p_3'.
\end{aligned}
$$

We define $n(\gamma_1, p_2, p_3')$ and $n(\gamma_1, p_2, \gamma_3')$ by

$$
\begin{aligned}
\partial^{f_3} m_2(\gamma_1,p_2) - m_2(\partial^{f_1}\gamma_1,p_2) &\pm m_2(\gamma_1,\partial^{f_2}p_2)\\
&= \sum_{p_3'} n(\gamma_1,p_2,p_3')p_3' + \sum_{\gamma_3'} n(\gamma_1,p_2,\gamma_3')\gamma_3'.
\end{aligned}
$$

Then

$$
\begin{aligned}
n(\gamma_1,p_2,p_3') = & \sum_{\delta_1,p_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3)\sharp\mathcal{M}(p_3,p_3') \\
& + \sum_{\delta_1',\delta_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1')\sharp\mathcal{M}_N(\delta_1',\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3') \\
& + \sum_{\delta_1,p_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)\sharp\mathcal{M}(p_1,p_2,p_3') \\
& + \sum_{\delta_1,p_2'} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(p_2,p_2')\sharp\mathcal{M}(\delta_1,p_2',p_3') \\
& + \sum_{\delta_1,\gamma_2,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(p_2,\gamma_2)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3') \\
& + \sum_{\delta_1,\gamma_2,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(p_2,\gamma_2)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3') \\
\overset{\textbf{(i)}}{=} & \sum_{\delta_1} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\partial\overline{\mathcal{M}}(\delta_1,p_2,p_3') \\
= & \ 0.
\end{aligned}
$$

Note that we use Theorem 4.3 **(i)** at $\overset{\textbf{(i)}}{=}$. Hence $n(\gamma_1,p_2,p_3') = 0$. Moreover, $n(\gamma_1,p_2,\gamma_3') = 0$. Hence we obtain $\partial^{f_3}m_2(\gamma_1,p_2) = m_2(\partial^{f_1}\gamma_1,p_2) \pm m_2(\gamma_1,\partial^{f_2}p_2)$. □

Theorem 4.7. *For positive boundary critical points γ_1,γ_2 of $f_{1\partial M}, f_{2\partial M}$, respectively, we obtain the Leibniz rule: (We omit the sign convention.)*

$$\partial^{f_3}m_2(\gamma_1,\gamma_2) = m_2(\partial^{f_1}\gamma_1,\gamma_2) \pm m_2(\gamma_1,\partial^{f_2}\gamma_2).$$

Proof. First we calculate $\partial^{f_3}m_2(\gamma_1,\gamma_2)$.

$$
\begin{aligned}
\partial^{f_3}m_2(\gamma_1,\gamma_2) = \partial^{f_3} \Bigg\{ & \sum_{p_3,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)p_3 \\
& + \sum_{p_3,\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3)p_3 \Bigg\} \\
= & \sum_{p_3',p_3,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)\mathcal{M}(p_3,p_3')p_3' \\
& + \sum_{\gamma_3,p_3,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)\mathcal{M}(p_3,\gamma_3)\gamma_3 \qquad \text{(R)} \\
& + \sum_{p_3',p_3,\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3)\sharp\mathcal{M}(p_3,p_3')p_3' \\
& + \sum_{\gamma_3,p_3,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3)\sharp\mathcal{M}(p_3,\gamma_3)\gamma_3. \qquad \text{(S)}
\end{aligned}
$$

Note that the line (R) is equal to 0 since there is no broken negative gradient trajectory from a negative boundary critical point δ_3 to a positive boundary critical point γ_3, and similarly, the line (S) is equal to 0 since there is no broken negative gradient trajectory from a negative boundary critical point δ_1 or δ_2 to a positive boundary critical point γ_3.

Next we calculate $m_2(\partial^{f_1}\gamma_1,\gamma_2)$ and $m_2(\gamma_1,\partial^{f_2}\gamma_2)$.

$$
\begin{aligned}
&m_2(\partial^{f_1}\gamma_1,\gamma_2)\\
&= m_2\left(\sum_{\gamma_1'}\sharp\mathcal{M}_N(\gamma_1,\gamma_1')\gamma_1' + \sum_{\delta_1,p_1}\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)p_1,\gamma_2\right)\\
&= \sum_{\gamma_1',p_3',\delta_1,\delta_3}\sharp\mathcal{M}_N(\gamma_1,\gamma_1')\sharp\mathcal{M}_N(\gamma_1',\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3')p_3'\\
&\quad+\sum_{\gamma_1',p_3',\delta_1,\delta_2}\sharp\mathcal{M}_N(\gamma_1,\gamma_1')\sharp\mathcal{M}_N(\gamma_1',\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3')p_3'\\
&\quad+\sum_{\delta_1,p_1,p_3',\gamma_1',\delta_3}\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)\sharp\mathcal{M}(p_1,\gamma_1')\sharp\mathcal{M}_N(\gamma_1',\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3')p_3' \qquad \text{(T)}\\
&\quad+\sum_{\delta_1,p_1,p_3',\delta_2}\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p_3')p_3'\\
&\quad+\sum_{\delta_1,p_1,\gamma_3,\gamma_1'}\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)\sharp\mathcal{M}(p_1,\gamma_1')\sharp\mathcal{M}_N(\gamma_1',\gamma_2,\gamma_3)\gamma_3. \qquad \text{(U)}
\end{aligned}
$$

Note that the line (T) is equal to 0 since there is no broken negative gradient trajectory from a negative boundary critical point δ_1 to a positive boundary critical point γ_1', and similarly, the line (U) is equal to 0 since there is no broken negative gradient trajectory from a negative boundary critical point δ_1 to a positive boundary critical point γ_1'. Moreover,

$$
\begin{aligned}
&m_2(\gamma_1,\partial^{f_2}\gamma_2)\\
&= m_2\left(\gamma_1,\sum_{\gamma_2'}\sharp\mathcal{M}_N(\gamma_2,\gamma_2')\gamma_2' + \sum_{\delta_2,p_2}\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_2,p_2)p_2\right)\\
&= \sum_{\gamma_2',p_3',\delta_1,\delta_3}\sharp\mathcal{M}_N(\gamma_2,\gamma_2')\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2',\delta_3)\sharp\mathcal{M}(\delta_3,p_3')p_3'\\
&\quad+\sum_{\gamma_2',p_3',\delta_1,\delta_2}\sharp\mathcal{M}_N(\gamma_2,\gamma_2')\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2',\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3')p_3'\\
&\quad+\sum_{\delta_2,p_2,p_3',\delta_1}\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_2,p_2)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p_3')p_3'.
\end{aligned}
$$

We define $n(\gamma_1,\gamma_2,p_3')$ and $n(\gamma_1,\gamma_2,\gamma_3')$ by

$$
\begin{aligned}
\partial^{f_3}m_2(\gamma_1,\gamma_2)&-m_2(\partial^{f_1}\gamma_1,\gamma_2)\pm m_2(\gamma_1,\partial^{f_2}\gamma_2)\\
&=\sum_{p_3'}n(\gamma_1,\gamma_2,p_3')p_3' + \sum_{\gamma_3'}n(\gamma_1,\gamma_2,\gamma_3')\gamma_3'.
\end{aligned}
$$

Then

$$n(\gamma_1,\gamma_2,p'_3) = \sum_{p_3,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p_3)\mathcal{M}(p_3,p'_3) \tag{V}$$

$$+ \sum_{p_3,\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p_3)\sharp\mathcal{M}(p_3,p'_3) \tag{W}$$

$$+ \sum_{\gamma'_1,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\gamma'_1)\sharp\mathcal{M}_N(\gamma'_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p'_3) \tag{X}$$

$$+ \sum_{\gamma'_1,\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\gamma'_1)\sharp\mathcal{M}_N(\gamma'_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p'_3) \tag{Y}$$

$$+ \sum_{\delta_1,p_1,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(p_1,\delta_2,p'_3) \tag{Z}$$

$$+ \sum_{\gamma'_2,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_2,\gamma'_2)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma'_2,\delta_3)\sharp\mathcal{M}(\delta_3,p'_3) \tag{A'}$$

$$+ \sum_{\gamma'_2,\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_2,\gamma'_2)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma'_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p'_3) \tag{B'}$$

$$+ \sum_{\delta_2,p_2,\delta_1} \sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_2,p_2)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}(\delta_1,p_2,p'_3). \tag{C'}$$

By Theorem 4.1 **(e)**, the line (Y) is equal to

$$\sum_{\delta'_1,\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta'_1)\sharp\mathcal{M}_N(\delta'_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p'_3), \tag{D'}$$

and similarly, the line (B') is equal to

$$\sum_{\delta'_2,\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_2,\delta'_2)\sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta'_2,\delta_2)\sharp\mathcal{M}(\delta_1,\delta_2,p'_3). \tag{E'}$$

Then, by Theorem 4.3 **(k)**, the sum of the lines (W), (D'), (Z), (E') and (C') is equal to

$$\sum_{\delta_1,\delta_2} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\gamma_2,\delta_2)\left\{\sharp\partial\overline{\mathcal{M}}(\delta_1,\delta_2,p'_3) + \sum_{\delta_3}\sharp\mathcal{M}_N(\delta_1,\delta_2,\delta_3)\sharp\mathcal{M}(\delta_3,p'_3)\right\}. \tag{F'}$$

Note that $\sharp\partial\overline{\mathcal{M}}(\delta_1,\delta_2,p'_3)$ is equal to 0.

By Theorem 4.1 **(c)**, the line (V) is equal to

$$\sum_{\delta_1,\delta'_3,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}_N(\delta_3,\delta'_3)\mathcal{M}(\delta'_3,p'_3), \tag{G'}$$

and, by Theorem 4.1 **(e)**, the line (X) is equal to

$$\sum_{\delta'_1,\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta'_1)\sharp\mathcal{M}_N(\delta'_1,\delta_1)\sharp\mathcal{M}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p'_3). \tag{H'}$$

Then, by Theorem 4.3 **(l)**, the sum of the lines (A'), (F'), (G') and (H') is equal to

$$\sum_{\delta_1,\delta_3} \sharp\mathcal{M}_N(\gamma_1,\delta_1)\sharp\partial\overline{\mathcal{M}}_N(\delta_1,\gamma_2,\delta_3)\sharp\mathcal{M}(\delta_3,p'_3) = 0.$$

Hence $n(\gamma_1, \gamma_2, p_3') = 0$. Moreover, $n(\gamma_1, \gamma_2, \gamma_3') = 0$. Therefore, $\partial^{f_3} m_2(\gamma_1, \gamma_2) = m_2(\partial^{f_1}\gamma_1, \gamma_2) \pm m_2(\gamma_1, \partial^{f_2}\gamma_2)$. $\square$

At last, we finish proving the Leibniz rules in terms of gradient trees!

Theorem 4.8. *We denote by $\partial^{f_1}, \partial^{f_2}$ and ∂^{f_3} the boundary operators of Morse complex for f_1, f_2 and f_3, respectively. Then we obtain the Leibniz rule: (We omit the sign convention.)*

$$\partial^{f_3} m_2(*_1 \otimes *_2) = m_2(\partial^{f_1} *_1 \otimes *_2) \pm m_2(*_1 \otimes \partial^{f_2} *_2),$$

where $*_i$ is an interior critical point of f_i or a positive boundary critical point of $f_{i\partial M}$, for $i = 1, 2$.

There is a remark about other related works; In [2] J. Bloom also studied product structures on Morse homology of manifolds with boundary; In fact he studied some A_∞ structure on Morse homology of manifolds with boundary, and he applied his A_∞ structures to Seiberg–Witten Floer theory.

5. Product structures on Floer homology

In this section, we define product structures on Floer homology of Lagrangian submanifolds with Legendrian end in a symplectic manifold with concave end, and observe the Leibniz rules on the chain level. But, before the product structures, we briefly recall the Floer homology, see [1].

Let M be a non-compact symplectic manifold with symplectic form ω, and N a compact contact manifold with contact form λ. Suppose that we have a compact subset $K \subset M$ such that $M \setminus K$ is diffeomorphic to $(-\infty, 0) \times N$. Moreover, we assume that $\omega = d(e^t\lambda)$ on $(-\infty, 0) \times N$, where t is the standard coordinate on the first factor. We call $M \setminus K = (-\infty, 0) \times N \subset M$ a concave end of M. We denote by R the Reeb vector field of (N, λ), and by ξ the contact distribution of (N, λ). Let $(\mathbb{R} \times N, d(e^t\lambda))$ be the symplectization of (N, λ). Note that we may have compatible almost complex structures J on $\mathbb{R} \times N$ such that $J\frac{\partial}{\partial t} = R$ and $J\xi = \xi$, and we also have compatible almost complex structures J on M, we use the same notation, such that the restriction of J on the concave end $(-\infty, 0) \times N$ satisfies $J\frac{\partial}{\partial t} = R$ and $J\xi = \xi$.

Let Λ_0, Λ_1 be a Legendrian submanifolds in N. We call a map $\gamma : [0, T] \to N$ a *positive* Reeb chord if $\dot{\gamma} = R \circ \gamma$ and $\gamma(0) \in \Lambda_1$ and $\gamma(T) \in \Lambda_0$, and similarly we call a map $\delta : [0, T] \to N$ a *negative* Reeb chord if $\dot{\delta} = R \circ \delta$ and $\delta(0) \in \Lambda_0$ and $\delta(T) \in \Lambda_1$. For each positive Reeb chord $\gamma : [0, T] \to N$ with $\gamma(0) \in \Lambda_1$ and $\gamma(T) \in \Lambda_0$, we assume that $d\phi_T(T_{\gamma(0)}\Lambda_1)$ and $T_{\gamma(T)}\Lambda_0$ intersect transversely in $\xi_{\gamma(T)}$, where $\phi_t : N \to N$ is the isotopy generated by the Reeb vector field, and similarly we also assume that $d\phi_T(T_{\delta(0)}\Lambda_0)$ and $T_{\delta(T)}\Lambda_1$ intersect transversely in $\xi_{\delta(T)}$, for each negative Reeb chord $\delta : [0, T] \to N$ with $\delta(0) \in \Lambda_0$ and $\delta(T) \in \Lambda_1$. Note that, once we have such transversality condition, Reeb chords are isolated. Let L_0 and L_1 be transversely intersecting Lagrangian submanifolds in M such that $L_0|_{(-\infty,0)\times N} = (-\infty, 0) \times \Lambda_0$ and $L_1|_{(-\infty,0)\times N} = (-\infty, 0) \times \Lambda_1$.

In this section, we always use the notation p, p' for intersection points of $L_0 \cap L_1$, γ, γ' for positive Reeb chords, and δ, δ' for negative Reeb chords.

We define the moduli spaces of pseudoholomorphic strips. For $p, p' \in L_0 \cap L_1$, we denote by $\mathcal{M}(p, p')$ the set of unparameterized pseudoholomorphic maps $u : \mathbb{R} \times [0, 1] \to M$ such that

- $du \circ i = J \circ du$, where i is the standard complex structure on $\mathbb{R} \times [0,1]$;
- $u(\mathbb{R}, 0) \subset L_0$ and $u(\mathbb{R}, 1) \subset L_1$; and
- $\lim_{t\to-\infty} u(t, [0,1]) = p$ and $\lim_{t\to\infty} u(t, [0,1]) = p'$.

For $p \in L_0 \cap L_1$ and a positive Reeb chord $\gamma : [0,T] \to N$, we denote by $\mathcal{M}(p, \gamma)$ the set of unparameterized pseudoholomorphic maps $u : \mathbb{R} \times [0,1] \to M$ such that

- $du \circ i = J \circ du$;
- $u(\mathbb{R}, 0) \subset L_0$ and $u(\mathbb{R}, 1) \subset L_1$;
- $\lim_{t\to-\infty} u([0,1], t) = p$; and
- For large $t > 0$, $u(t, [0,1]) \subset (-\infty, 0) \times N$ and $\lim_{t\to\infty} \pi_1 \circ u(t,s) = -\infty$ and $\lim_{t\to\infty} \pi_2 \circ u(t,s) = \gamma(T(1-s))$,

where $\pi_1 : (-\infty, 0) \times N \to (-\infty, 0)$ is the projection on the first factor and $\pi_2 : (-\infty, 0) \times N \to N$ is the projection on the second factor. Similarly we define $\mathcal{M}(\delta, p)$ and $\mathcal{M}(\delta, \gamma)$, for a negative Reeb chord $\delta : [0,T] \to N$. Next, for positive Reeb chords $\gamma : [0,T] \to N$ and $\gamma' : [0,T'] \to N$, we denote by $\mathcal{M}_N(\gamma, \gamma')$ the set of unparameterized pseudoholomorphic maps $u : \mathbb{R} \times [0,1] \to \mathbb{R} \times N$ up to the $\mathbb{R}$-translation of $\mathbb{R} \times N$ such that

- $du \circ i = J \circ du$;
- $u(\mathbb{R}, 0) \subset \mathbb{R} \times \Lambda_0$ and $u(\mathbb{R}, 1) \subset \mathbb{R} \times \Lambda_1$;
- $\lim_{t\to-\infty} \pi_1 \circ u = \infty$ and $\lim_{t\to-\infty} \pi_2 \circ u(t,s) = \gamma(T(1-s))$; and
- $\lim_{t\to\infty} \pi_1 \circ u = -\infty$ and $\lim_{t\to\infty} \pi_2 \circ u(t,s) = \gamma'(T'(1-s))$,

where $\pi_1 : \mathbb{R} \times N \to \mathbb{R}$ is the projection on the first factor and $\pi_2 : \mathbb{R} \times N \to N$ is the projection on the second factor. Similarly, for a positive Reeb chord $\gamma : [0,T] \to N$ and a negative Reeb chord $\delta : [0,T'] \to N$, we denote by $\mathcal{M}_N(\gamma, \delta)$ the set of unparameterized pseudoholomorphic maps $u : \mathbb{R} \times [0,1] \to \mathbb{R} \times N$ up to the $\mathbb{R}$-translation of $\mathbb{R} \times N$ such that

- $du \circ i = J \circ du$;
- $u(\mathbb{R}, 0) \subset \mathbb{R} \times \Lambda_0$ and $u(\mathbb{R}), 1 \subset \mathbb{R} \times \Lambda_1$;
- $\lim_{t\to-\infty} \pi_1 \circ u = \infty$ and $\lim_{t\to-\infty} \pi_2 \circ u(t,s) = \gamma(T(1-s))$; and
- $\lim_{t\to\infty} \pi_1 \circ u = \infty$ and $\lim_{t\to\infty} \pi_2 \circ u(t,s) = \delta(T's)$.

For a negative Reeb chords $\delta : [0,T] \to N, \delta' : [0,T'] \to N$, we denote by $\mathcal{M}_N(\delta, \delta')$ the set of unparameterized pseudoholomorphic maps $u : \mathbb{R} \times [0,1] \to \mathbb{R} \times N$ up to the $\mathbb{R}$-translation of $\mathbb{R} \times N$ such that

- $du \circ i = J \circ du$;
- $u(\mathbb{R}, 0) \subset \mathbb{R} \times \Lambda_0$ and $u(\mathbb{R}, 1) \subset \mathbb{R} \times \Lambda_1$;
- $\lim_{t\to-\infty} \pi_1 \circ u = -\infty$ and $\lim_{t\to-\infty} \pi_2 \circ u(t,s) = \delta(Ts)$; and
- $\lim_{t\to\infty} \pi_1 \circ u = \infty$ and $\lim_{t\to\infty} \pi_2 \circ u(t,s) = \delta'(T's)$.

We remark that, for a negative Reeb chord $\delta : [0,T] \to N$ and a positive Reeb chord $\gamma : [0,T'] \to N$, there is no pseudoholomorphic map $u : \mathbb{R} \times [0,1] \to \mathbb{R} \times N$ such that

- $du \circ i = J \circ du$;
- $u(\mathbb{R}, 0) \subset \mathbb{R} \times \Lambda_0$ and $u(\mathbb{R}, 1) \subset \mathbb{R} \times \Lambda_1$;
- $\lim_{t\to-\infty} \pi_1 \circ u = -\infty$ and $\lim_{t\to-\infty} \pi_2 \circ u(t,s) = \delta(Ts)$; and
- $\lim_{t\to\infty} \pi_1 \circ u = -\infty$ and $\lim_{t\to\infty} \pi_2 \circ u(t,s) = \gamma(T'(1-s))$

because of the maximal principle. Hence $\mathcal{M}_N(\delta, \gamma) = \emptyset$.

Now we observe these moduli spaces. In this paper we call the following pseudoholomorphic maps bubbles.

- $u : D := \{z \in \mathbb{C} : |z| \leq 1\} \to M$ such that $u(\partial D) \subset L_0$ or $u(\partial D) \subset L_1$, and $\int_D u^*\omega < \infty$;
- $u : \mathbb{H} := \{z = x + iy \in \mathbb{C} : y \geq 0\} \to \mathbb{R} \times N$ such that $u(\partial \mathbb{H}) \subset \mathbb{R} \times \Lambda_0$ or $u(\partial \mathbb{H}) \subset \Lambda_1$, and $\int_{\mathbb{H}} u^*\lambda < \infty$; and
- $u : \mathbb{C} \to \mathbb{R} \times N$ such that $\int_{\mathbb{C}} u^*\lambda < \infty$.

To define our Floer homology, we have to avoid bubbles as above.

Theorem 5.1. *Suppose no bubble and moduli spaces are transversal. For simplicity, we assume that the dimension of the moduli spaces are independent of the homotopy types of pseudoholomorphic maps.*

(a) *$\mathcal{M}(p,p')$ is a finite dimensional smooth manifold. If $\dim \mathcal{M}(p,p') = 0$, then $\mathcal{M}(p,p')$ is compact. If $\dim \mathcal{M}(p,p') = 1$, then $\mathcal{M}(p,p')$ can be compactified into $\overline{\mathcal{M}}(p,p')$, whose boundary is*

$$\partial \overline{\mathcal{M}}(p,p') = \bigcup_{p''} \mathcal{M}(p,p'') \times \mathcal{M}(p'',p') \cup \bigcup_{\gamma,\delta} \mathcal{M}(p,\gamma) \times \mathcal{M}_N(\gamma,\delta) \times \mathcal{M}(\delta,p'),$$

where $p'' \in L_0 \cap L_1$, γ is a positive Reeb chord, and δ is a negative Reeb chord.

(b) *$\mathcal{M}(p,\gamma)$ is a finite dimensional smooth manifold. If $\dim \mathcal{M}(p,\gamma) = 0$, then $\mathcal{M}(p,\gamma)$ is compact. If $\dim \mathcal{M}(p,\gamma) = 1$, then $\mathcal{M}(p,\gamma)$ can be compactified into $\overline{\mathcal{M}}(p,\gamma)$, whose boundary is*

$$\begin{aligned}\partial \overline{\mathcal{M}}(p,\gamma) = & \bigcup_{p'} \mathcal{M}(p,p') \times \mathcal{M}(p',\gamma) \cup \bigcup_{\gamma'} \mathcal{M}(p,\gamma') \times \mathcal{M}_N(\gamma',\gamma), \\ & \cup \bigcup_{\gamma',\delta} \mathcal{M}(p,\gamma') \times \mathcal{M}_N(\gamma',\delta) \times \mathcal{M}(\delta,\gamma),\end{aligned}$$

where $p' \in L_0 \cap L_1$, γ' is a positive Reeb chord, and δ is a negative Reeb chord.

(c) *$\mathcal{M}(\delta,p)$ is a finite dimensional smooth manifold. If $\dim \mathcal{M}(\delta,p) = 0$, then $\mathcal{M}(\delta,p)$ is compact. If $\dim \mathcal{M}(\delta,p) = 1$, then $\mathcal{M}(\delta,p)$ can be compactified into $\overline{\mathcal{M}}(\delta,p)$, whose boundary is*

$$\begin{aligned}\partial \overline{\mathcal{M}}(\delta,p) = & \bigcup_{p'} \mathcal{M}(\delta,p') \times \mathcal{M}(p',p) \cup \bigcup_{\delta'} \mathcal{M}_N(\delta,\delta') \times \mathcal{M}(\delta',p), \\ & \cup \bigcup_{\gamma,\delta'} \mathcal{M}(\delta,\gamma) \times \mathcal{M}_N(\gamma,\delta') \times \mathcal{M}(\delta',p),\end{aligned}$$

where $p' \in L_0 \cap L_1$, γ is a positive Reeb chord, and δ' is a negative Reeb chord.

(d) *$\mathcal{M}_N(\gamma,\gamma')$ is a finite dimensional smooth manifold. If $\dim \mathcal{M}_N(\gamma,\gamma') = 0$, then $\mathcal{M}_N(\gamma,\gamma')$ is compact. If $\dim \mathcal{M}_N(\gamma,\gamma') = 1$, then $\mathcal{M}_N(\gamma,\gamma')$ can be compactified into $\overline{\mathcal{M}}_N(\gamma,\gamma')$, whose boundary is*

$$\partial \overline{\mathcal{M}}_N(\gamma,\gamma') = \bigcup_{\gamma''} \mathcal{M}_N(\gamma,\gamma'') \times \mathcal{M}_N(\gamma'',\gamma'),$$

where γ'' is a positive Reeb chord.

(e) *$\mathcal{M}_N(\gamma,\delta)$ is a finite smooth manifold. If $\dim \mathcal{M}_N(\gamma,\delta) = 0$, then $\mathcal{M}_N(\gamma,\delta)$ is compact. If $\dim \mathcal{M}_N(\gamma,\delta) = 1$, then $\mathcal{M}_N(\gamma,\delta)$ can be compactified into $\overline{\mathcal{M}}_N(\gamma,\delta)$, whose boundary is*

$$\partial \overline{\mathcal{M}}_N(\gamma,\delta) = \bigcup_{\gamma'} \mathcal{M}_N(\gamma,\gamma') \times \mathcal{M}_N(\gamma',\delta) \cup \bigcup_{\delta'} \mathcal{M}_N(\gamma,\delta') \times \mathcal{M}_N(\delta',\delta),$$

where γ' is a positive Reeb chord and δ' is a negative Reeb chord.

(f) *$\mathcal{M}(\delta,\gamma)$ is a finite smooth manifold. If $\dim\mathcal{M}(\delta,\gamma)=0$, then $\mathcal{M}(\delta,\gamma)$ is compact. If $\dim\mathcal{M}(\delta,\gamma)=1$, then $\mathcal{M}(\delta,\gamma)$ can be compactified into $\overline{\mathcal{M}}(\delta,\gamma)$, whose boundary is*

$$\begin{aligned}\partial\overline{\mathcal{M}}(\delta,\gamma)=&\bigcup_p\mathcal{M}(\delta,p)\times\mathcal{M}(p,\gamma)\cup\bigcup_{\gamma',\delta'}\mathcal{M}(\delta,\gamma')\times\mathcal{M}_N(\gamma',\delta')\times\mathcal{M}(\delta',\gamma)\\&\cup\bigcup_{\delta'}\mathcal{M}_N(\delta,\delta')\times\mathcal{M}(\delta',\gamma)\cup\bigcup_{\gamma'}\mathcal{M}(\delta,\gamma')\times\mathcal{M}_N(\gamma',\gamma),\end{aligned}$$

where $p\in L_0\cap L_1$, γ' is a positive Reeb chord, and δ' is a negative Reeb chord.

We omit the proof of Theorem 5.1. Note that we may list every boundary components of 1-dimensional moduli spaces in Theorem 5.1 without omission by chasing intersection points and Reeb chords so that we obtain 1-dimensional moduli spaces after gluing pseudoholomorphic strips. Note that, in Morse homology, there is no broken negative trajectory from a negative boundary critical point to a positive boundary critical point. But, in Floer case, we have broken pseudoholomorphic strips in M from a negative Reeb chord to a positive Reeb chord.

We define

$$C(L_0,L_1):=\bigoplus_{p\in L_0\cap L_1}\mathbb{Z}_2p\oplus\bigoplus_{\gamma:\Lambda_1\to\Lambda_0}\mathbb{Z}_2\gamma,$$

where γ is a positive Reeb chord, and define a linear map $\partial:C(L_0,L_1)\to C(L_0,L_1)$ by

$$\begin{aligned}\partial p:=&\sum_{p'}\sharp\mathcal{M}(p,p')p'+\sum_{\gamma'}\sharp\mathcal{M}(p,\gamma')\gamma',\\ \partial\gamma:=&\sum_{\gamma'}\sharp\mathcal{M}_N(\gamma,\gamma')\gamma'+\sum_{\delta,\gamma'}\sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,\gamma')\gamma'+\sum_{\delta,p'}\sharp\mathcal{M}_N(\gamma,\delta)\sharp\mathcal{M}(\delta,p')p',\end{aligned}$$

where each moduli space is a 0-dimensional component. Note that the definition of ∂ is slightly different from the boundary operator of Morse complex.

As in the Morse case, Theorem 4.2, we can prove the following theorem by observing the boundary of 1-dimensional components of the moduli spaces of pseudoholomrophic strips in Theorem 5.1. We omit the proof.

Theorem 5.2. *Suppose no bubble, and $\partial\circ\partial=0$.*

We obtain a chain complex $(C(L_0,L_1),\partial)$, and its homology is our Floer homology.

Next we observe the Leibniz rules.

Let M be a symplectic manifold with concave end as before, and L_i a Lagrangian submanifold with Legendrian end $(-\infty,0)\times\Lambda_i$ in M, for $i=0,1,2$. We assume that each pair L_i and L_j, $i\neq j$, intersect transversely and the Reeb chords are isolated as before. In this case we call a map $\gamma_{ij}:[0,T]\to N$ a positive Reeb chord for (L_i,L_j) if $\dot\gamma_{ij}=R\circ\gamma_{ij}$, and $\gamma_{ij}(0)\in\Lambda_j$ and $\gamma_{ij}(T)\in\Lambda_i$, and similarly we call a map $\delta_{ij}:[0,T]\to N$ a negative Reeb chord for (L_i,L_j) if $\dot\delta=R\circ\delta$, and $\delta(0)\in\Lambda_i$ and $\delta_{ij}(T)\in\Lambda_j$.

Let $D:=\{z\in\mathbb{C}:|z|\leq1\}$, and we take $z_0,z_1,z_2\in\partial D$ in clockwise order. We define $\Sigma:=D\setminus\{z_0,z_1,z_2\}$, and we denote by $l_0\subset\partial\Sigma$ the open arc between z_0 and z_1, by $l_1\subset\partial\Sigma$ the open arc between z_1 and z_2, and by $l_2\subset\partial\Sigma$ the open arc between z_2 and z_0. For $i=0,1,2$, we may take an neighborhood $U_i\subset D$ of z_i such that there are biholomorphic maps $\phi_i:(-\infty,0)\times[0,1]\to U_i\setminus\{z_i\}$

with $\lim_{t\to-\infty}\phi_i(t,s)=z_i$, for $i=1,2$, and $\phi_0:(0,\infty)\times[0,1]\to U_0\setminus\{z_0\}$ with $\lim_{t\to\infty}\phi_0(t,s)=z_0$.

We define the moduli spaces of pseudoholomorphic triangles. For $p_{01}\in L_0\cap L_1, p_{12}\in L_1\cap L_2, p_{02}\in L_0\cap L_2$, we denote by $\mathcal{M}(p_{01},p_{12},p_{02})$ the set of pseudoholomorphic maps $u:\Sigma\to M$ such that

- $du\circ i=J\circ du$, where i is the standard complex structure on Σ;
- $u(l_0)\subset L_0, u(l_1)\subset L_1$ and $u(l_2)\subset L_2$; and
- $\lim_{t\to-\infty}u\circ\phi_1(t,s)=p_{01}$, $\lim_{t\to-\infty}u\circ\phi_2(t,s)=p_{12}$ and $\lim_{t\to\infty}u\circ\phi_0(t,s)=p_{02}$.

For $p_{01}\in L_0\cap L_1, p_{12}\in L_1\cap L_2$ and a positive Reeb chord $\gamma_{02}:[0,T]\to N$ for (L_0,L_2), we denote by $\mathcal{M}(p_{01},p_{12},\gamma_{02})$ the set of pseudoholomorphic maps $u:\Sigma\to M$ such that

- $du\circ i=J\circ du$;
- $u(l_0)\subset L_0, u(l_1)\subset L_1$ and $u(l_2)\subset L_2$;
- $\lim_{t\to-\infty}u\circ\phi_1(t,s)=p_{01}$ and $\lim_{t\to-\infty}u\circ\phi_2(t,s)=p_{12}$; and
- For large $t>0$, $u\circ\phi_0(t,[0,1])\subset(-\infty,0)\times N$ and $\lim_{t\to\infty}\pi_1\circ u\circ\phi_0(t,s)=-\infty$ and $\lim_{t\to\infty}\pi_2\circ u\circ\phi_0(t,s)=\gamma_{02}(T(1-s))$,

where $\pi_1:(-\infty,0)\times N\to(-\infty,0)$ is the projection on the first factor and $\pi_2:(-\infty,0)\times N\to N$ is the projection on the second factor. For a negative Reeb chord $\delta_{01}:[0,T]\to N$ for (L_0,L_1) and $p_{12}\in L_1\cap L_2, p_{02}\in L_0\cap L_2$, we denote by $\mathcal{M}(\delta_{01},p_{12},p_{02})$ the set of pseudoholomorphic maps $u:\Sigma\to M$ such that

- $du\circ i=J\circ du$;
- $u(l_0)\subset L_0, u(l_1)\subset L_1$ and $u(l_2)\subset L_2$;
- For large $-t>0$, $u\circ\phi_1(t,[0,1])\subset(-\infty,0)\times N$ and $\lim_{t\to-\infty}\pi_1\circ u\circ\phi_1(t,s)=-\infty$ and $\lim_{t\to-\infty}\pi_2\circ u\circ\phi_1(t,s)=\delta_{01}(Ts)$; and
- $\lim_{t\to-\infty}u\circ\phi_2(t,s)=p_{12}$ and $\lim_{t\to\infty}u\circ\phi_0(t,s)=p_{02}$.

Similarly, we define $\mathcal{M}(\delta_{01},p_{12},\gamma_{02}),\mathcal{M}(p_{01},\delta_{12},p_{02}),\mathcal{M}(p_{01},\delta_{12},\gamma_{02}),\mathcal{M}(\delta_{01},\delta_{12},p_{02})$ and $\mathcal{M}(\delta_{01},\delta_{12},\gamma_{02})$. For positive Reeb chords $\gamma_{ij}:[0,T_{ij}]\to N$ for (L_i,L_j), we denote by $\mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02})$ the set of pseudoholomorphic maps $u:\Sigma\to\mathbb{R}\times N$ up to the $\mathbb{R}$-translation of $\mathbb{R}\times N$ such that

- $du\circ i=J\circ u$;
- $u(l_0)\subset\mathbb{R}\times\Lambda_0, u(l_1)\subset\mathbb{R}\times\Lambda_1$ and $u(l_2)\subset\mathbb{R}\times\Lambda_2$;
- $\lim_{t\to-\infty}\pi_1\circ u\circ\phi_1=\infty$ and $\lim_{t\to-\infty}\pi_2\circ u\circ\phi_1(t,s)=\gamma_{01}(T_{01}(1-s))$;
- $\lim_{t\to-\infty}\pi_1\circ u\circ\phi_2=\infty$ and $\lim_{t\to-\infty}\pi_2\circ u\circ\phi_2(t,s)=\gamma_{12}(T_{12}(1-s))$; and
- $\lim_{t\to\infty}\pi_1\circ u\circ\phi_0=-\infty$ and $\lim_{t\to\infty}\pi_2\circ u\circ\phi_0(t,s)=\gamma_{02}(T_{02}(1-s))$,

where $\pi_1:\mathbb{R}\times N\to\mathbb{R}$ is the projection on the first factor and $\pi_2:\mathbb{R}\times N\to N$ is the projection on the second factor. For positive Reeb chords $\gamma_{ij}:[0,T_{ij}]\to N$ for (L_i,L_j) and a negative Reeb chord $\delta_{02}:[0,T_{02}]\to N$, we denote by $\mathcal{M}_N(\gamma_{01},\gamma_{12},\delta_{02})$ the set of pseudoholomorphic maps $u:\Sigma\to\mathbb{R}\times N$ up to the $\mathbb{R}$-translation of $\mathbb{R}\times N$ such that

- $du\circ i=J\circ u$;
- $u(l_0)\subset\mathbb{R}\times\Lambda_0, u(l_1)\subset\mathbb{R}\times\Lambda_1$ and $u(l_2)\subset\mathbb{R}\times\Lambda_2$;
- $\lim_{t\to-\infty}\pi_1\circ u\circ\phi_1=\infty$ and $\lim_{t\to-\infty}\pi_2\circ u\circ\phi_1(t,s)=\gamma_{01}(T_{01}(1-s))$;
- $\lim_{t\to-\infty}\pi_1\circ u\circ\phi_2=\infty$ and $\lim_{t\to-\infty}\pi_2\circ u\circ\phi_2(t,s)=\gamma_{12}(T_{12}(1-s))$; and
- $\lim_{t\to\infty}\pi_1\circ u\circ\phi_0=\infty$ and $\lim_{t\to\infty}\pi_2\circ u\circ\phi_0(t,s)=\delta_{02}(T_{02}s)$.

Similarly, we define $\mathcal{M}_N(\delta_{01},\gamma_{12},\gamma_{02}),\mathcal{M}_N(\delta_{01},\gamma_{12},\delta_{02}),\mathcal{M}_N(\gamma_{01},\delta_{12},\gamma_{02}),\mathcal{M}_N(\gamma_{01},\delta_{12},\delta_{02})$ and $\mathcal{M}_N(\delta_{01},\delta_{12},\delta_{02})$. We remark that, for negative Reeb chords $\delta_{ij}:[0,T_{ij}]\to N$ and a positive Reeb chord $\gamma_{02}:[0,T_{02}]\to N$, there is no pseudoholomorphic maps such that

- $du\circ i = J\circ u$;
- $u(l_0)\subset\mathbb{R}\times\Lambda_0, u(l_1)\subset\mathbb{R}\times\Lambda_1$ and $u(l_2)\subset\mathbb{R}\times\Lambda_2$;
- $\lim_{t\to-\infty}\pi_1\circ u\circ\phi_1=-\infty$ and $\lim_{t\to-\infty}\pi_2\circ u\circ\phi_1(t,s)=\delta_{01}(T_{01}s)$;
- $\lim_{t\to-\infty}\pi_1\circ u\circ\phi_2=-\infty$ and $\lim_{t\to-\infty}\pi_2\circ u\circ\phi_2(t,s)=\delta_{12}(T_{12}s)$; and
- $\lim_{t\to\infty}\pi_1\circ u\circ\phi_0=-\infty$ and $\lim_{t\to\infty}\pi_2\circ u\circ\phi_0(t,s)=\gamma_{02}(T_{12}(1-s))$

because of the maximal principle. Hence $\mathcal{M}_N(\delta_{01},\delta_{12},\gamma_{02})=\emptyset$.

Now we observe these moduli spaces. Note that we always use notation, for $i=0,1,2$,

- $p_{ij},p'_{ij},p''_{ij}\in L_i\cap L_j$;
- $\gamma_{ij},\gamma'_{ij},\gamma''_{ij}$ for positive Reeb chords for (L_i,L_j); and
- $\delta_{ij},\delta'_{ij},\delta''_{ij}$ for negative Reeb chords for (L_i,L_j).

Then we have the following theorem:

Theorem 5.3. *Suppose no bubble and moduli spaces are transversal. For simplicity, we assume that the dimension of the moduli spaces are independent of the homotopy types of pseudoholomorphic maps.*

(g) $\mathcal{M}(p_{01},p_{12},p_{02})$ *is a finite dimensional smooth manifold. If* $\dim\mathcal{M}(p_{01},p_{12},p_{02})=0$, *then* $\mathcal{M}(p_{01},p_{12},p_{02})$ *is compact. If* $\mathcal{M}(p_{01},p_{12},p_{02})=1$, *then* $\mathcal{M}(p_{01},p_{12},p_{02})$ *can be compactified into* $\overline{\mathcal{M}}(p_{01},p_{12},p_{02})$, *whose boundary is*

$$\begin{aligned}\partial\overline{\mathcal{M}}(p_{01},p_{12},p_{02}) =&\bigcup_{p'_{01}}\mathcal{M}(p_{01},p'_{01})\times\mathcal{M}(p'_{01},p_{12},p_{02})\\ &\cup\bigcup_{\gamma_{01},\delta_{01}}\mathcal{M}(p_{01},\gamma_{01})\times\mathcal{M}_N(\gamma_{01},\delta_{01})\times\mathcal{M}(\delta_{01},p_{12},p_{02})\\ &\cup\bigcup_{p'_{12}}\mathcal{M}(p_{12},p'_{12})\times\mathcal{M}(p_{01},p'_{12},p_{02})\\ &\cup\bigcup_{\gamma_{12},\delta_{12}}\mathcal{M}(p_{12},\gamma_{12})\times\mathcal{M}_N(\gamma_{12},\delta_{12})\times\mathcal{M}(p_{01},\delta_{12},p_{02})\\ &\cup\bigcup_{p'_{02}}\mathcal{M}(p_{01},p_{12},p'_{02})\times\mathcal{M}(p'_{02},p_{02})\\ &\cup\bigcup_{\gamma_{02},\delta_{02}}\mathcal{M}(p_{01},p_{12},\gamma_{02})\times\mathcal{M}_N(\gamma_{02},\delta_{02})\times\mathcal{M}(\delta_{02},p_{02})\\ &\cup\bigcup_{\gamma_{01},\gamma_{12},\delta_{02}}\mathcal{M}(p_{01},\gamma_{01})\times\mathcal{M}(p_{12},\gamma_{12})\times\mathcal{M}_N(\gamma_{01},\gamma_{12},\delta_{02})\times\mathcal{M}(\delta_{02},p_{02}).\end{aligned}$$

(h) $\mathcal{M}(p_{01},p_{12},\gamma_{02})$ *is a finite dimensional smooth manifold. If* $\dim\mathcal{M}(p_{01},p_{12},\gamma_{02})=0$, *then* $\mathcal{M}(p_{01},p_{12},\gamma_{02})$ *is compact. If* $\mathcal{M}(p_{01},p_{12},\gamma_{02})=1$, *then* $\mathcal{M}(p_{01},p_{12},\gamma_{02})$

can be compactified into $\overline{\mathcal{M}}(p_{01}, p_{12}, \gamma_{02})$*, whose boundary is*

$$\begin{aligned}
\partial\overline{\mathcal{M}}(p_{01}, p_{12}, \gamma_{02}) = &\bigcup_{p'_{01}} \mathcal{M}(p_{01}, p'_{01}) \times \mathcal{M}(p'_{01}, p_{12}, \gamma_{02}) \\
&\cup \bigcup_{\gamma_{01}, \delta_{01}} \mathcal{M}(p_{01}, \gamma_{01}) \times \mathcal{M}_N(\gamma_{01}, \delta_{01}) \times \mathcal{M}(\delta_{01}, p_{12}, \gamma_{02}) \\
&\cup \bigcup_{p'_{12}} \mathcal{M}(p_{12}, p'_{12}) \times \mathcal{M}(p_{01}, p'_{12}, \gamma_{02}) \\
&\cup \bigcup_{\gamma_{12}, \delta_{12}} \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{12}, \delta_{12}) \times \mathcal{M}(p_{01}, \delta_{12}, \gamma_{02}) \\
&\cup \bigcup_{p_{02}} \mathcal{M}(p_{01}, p_{12}, p_{02}) \times \mathcal{M}(p_{02}, \gamma_{02}) \\
&\cup \bigcup_{\gamma'_{02}} \mathcal{M}(p_{01}, p_{12}, \gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02}, \gamma_{02}) \\
&\cup \bigcup_{\gamma'_{02}, \delta_{02}} \mathcal{M}(p_{01}, p_{12}, \gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02}, \delta_{02}) \times \mathcal{M}(\delta_{02}, \gamma_{02}) \\
&\cup \bigcup_{\gamma_{01}, \gamma_{12}} \mathcal{M}(p_{01}, \gamma_{01}) \times \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{01}, \gamma_{12}, \gamma_{02}) \\
&\cup \bigcup_{\gamma_{01}, \gamma_{02}, \delta_{02}} \mathcal{M}(p_{01}, \gamma_{01}) \times \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{01}, \gamma_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, \gamma_{02}).
\end{aligned}$$

(i) $\mathcal{M}(\delta_{01}, p_{12}, p_{02})$ *is a finite dimensional smooth manifold. If* $\dim \mathcal{M}(\delta_{01}, p_{12}, p_{02}) = 0$*, then* $\mathcal{M}(\delta_{01}, p_{12}, p_{02})$ *is compact. If* $\mathcal{M}(\delta_{01}, p_{12}, p_{02}) = 1$*, then* $\mathcal{M}(\delta_{01}, p_{12}, p_{02})$ *can be compactified into* $\overline{\mathcal{M}}(\delta_{01}, p_{12}, p_{02})$*, whose boundary is*

$$\begin{aligned}
\partial\overline{\mathcal{M}}(\delta_{01}, p_{12}, p_{02}) = &\bigcup_{p'_{01}} \mathcal{M}(\delta_{01}, p'_{01}) \times \mathcal{M}(p'_{01}, p_{12}, p_{02}) \\
&\cup \bigcup_{\delta'_{01}} \mathcal{M}_N(\delta_{01}, \delta'_{01}) \times \mathcal{M}(\delta'_{01}, p_{12}, p_{02}) \\
&\cup \bigcup_{\gamma_{01}, \delta'_{01}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}_N(\gamma_{01}, \delta'_{01}) \times \mathcal{M}(\delta'_{01}, p_{12}, p_{02}) \\
&\cup \bigcup_{\gamma_{12}, \delta_{01}} \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\delta_{01}, \gamma_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}) \\
&\cup \bigcup_{p'_{12}} \mathcal{M}(p_{12}, p'_{12}) \times \mathcal{M}(\delta_{01}, p'_{12}, p_{02}) \\
&\cup \bigcup_{\gamma_{12}, \delta_{12}} \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{12}, \delta_{12}) \times \mathcal{M}(\delta_{01}, \delta_{12}, p_{02}) \\
&\cup \bigcup_{p'_{02}} \mathcal{M}(\delta_{01}, p_{12}, p_{02}) \times \mathcal{M}(p'_{02}, p_{02}) \\
&\cup \bigcup_{\gamma_{02}, \delta_{02}} \mathcal{M}(\delta_{01}, p_{12}, \gamma_{02}) \times \mathcal{M}_N(\gamma_{02}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}) \\
&\cup \bigcup_{\gamma_{01}, \gamma_{12}, \delta_{02}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{01}, \gamma_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}).
\end{aligned}$$

(j) $\mathcal{M}(\delta_{01}, p_{12}, \gamma_{02})$ *is a finite dimensional smooth manifold. If* $\dim \mathcal{M}(\delta_{01}, p_{12}, \gamma_{02}) = 0$*, then* $\mathcal{M}(\delta_{01}, p_{12}, \gamma_{02})$ *is compact. If* $\mathcal{M}(\delta_{01}, p_{12}, \gamma_{02}) = 1$*, then* $\mathcal{M}(\delta_{01}, p_{12}, \gamma_{02})$ *can be compactified into* $\overline{\mathcal{M}}(\delta_{01}, p_{12}, \gamma_{02})$*, whose boundary is*

$$
\begin{aligned}
\partial\overline{\mathcal{M}}(\delta_{01}, p_{12}, \gamma_{02}) = & \bigcup_{p_{01}} \mathcal{M}(\delta_{01}, p_{01}) \times \mathcal{M}(p_{01}, p_{12}, \gamma_{02}) \\
& \cup \bigcup_{\delta'_{01}} \mathcal{M}_N(\delta_{01}, \delta'_{01}) \times \mathcal{M}(\delta'_{01}, p_{12}, \gamma_{02}) \\
& \cup \bigcup_{\gamma_{01}, \delta'_{01}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}_N(\gamma_{01}, \delta'_{01}) \times \mathcal{M}(\delta'_{01}, p_{12}, \gamma_{02}) \\
& \cup \bigcup_{\gamma_{12}, \delta_{02}} \times \mathcal{M}(p_{12}, \gamma_{12}) \mathcal{M}_N(\delta_{01}, \gamma_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, \gamma_{02}) \\
& \cup \bigcup_{p'_{12}} \mathcal{M}(p_{12}, p'_{12}) \times \mathcal{M}(\delta_{01}, p'_{12}, p_{02}) \\
& \cup \bigcup_{\gamma_{12}, \delta_{12}} \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{12}, \delta_{12}) \times \mathcal{M}(\delta_{01}, \delta_{12}, p_{02}) \\
& \cup \bigcup_{p_{02}} \mathcal{M}(\delta_{01}, p_{12}, p_{02}) \times \mathcal{M}(p_{02}, \gamma_{02}) \\
& \cup \bigcup_{\gamma'_{02}} \mathcal{M}(\delta_{01}, p_{12}, \gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02}, \gamma_{02}) \\
& \cup \bigcup_{\gamma'_{02}, \delta_{02}} \mathcal{M}(\delta_{01}, p_{12}, \gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02}, \delta_{02}) \times \mathcal{M}(\delta_{02}, \gamma_{02}) \\
& \cup \bigcup_{\gamma_{01}, \gamma_{12}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{01}, \gamma_{12}, \gamma_{02}) \\
& \cup \bigcup_{\gamma_{12}} \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\delta_{01}, \gamma_{12}, \gamma_{02}) \\
& \cup \bigcup_{\gamma_{01}, \gamma_{12}, \delta_{02}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}(p_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{01}, \gamma_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, \gamma_{02}).
\end{aligned}
$$

(k) $\mathcal{M}(p_{01}, \delta_{12}, p_{02})$ *is a finite dimensional smooth manifold. If* $\dim \mathcal{M}(p_{01}, \delta_{12}, p_{02}) = 0$*, then* $\mathcal{M}(p_{01}, \delta_{12}, p_{02})$ *is compact. If* $\mathcal{M}(p_{01}, \delta_{12}, p_{02}) = 1$*, then* $\mathcal{M}(p_{01}, \delta_{12}, p_{02})$

can be compactified into $\overline{\mathcal{M}}(p_{01},\delta_{12},p_{02})$*, whose boundary is*

$$\begin{aligned}\partial\overline{\mathcal{M}}(p_{01},\delta_{12},p_{02}) =& \bigcup_{p'_{01}} \mathcal{M}(p_{01},p'_{01})\times\mathcal{M}(p'_{01},\delta_{12},p_{02})\\ &\cup \bigcup_{\gamma_{01},\delta_{01}} \mathcal{M}(p_{01},\gamma_{01})\times\mathcal{M}_N(\gamma_{01},\delta_{01})\times\mathcal{M}(\delta_{01},\delta_{12},p_{02})\\ &\cup \bigcup_{p_{12}} \mathcal{M}(\delta_{12},p_{12})\times\mathcal{M}(p_{01},p_{12},p_{02})\\ &\cup \bigcup_{\gamma_{12},\delta'_{12}} \mathcal{M}(\delta_{12},\gamma_{12})\times\mathcal{M}_N(\gamma_{12},\delta'_{12})\times\mathcal{M}(p_{01},\delta'_{12},p_{02})\\ &\cup \bigcup_{\delta'_{12}} \mathcal{M}_N(\delta_{12},\delta'_{12})\times\mathcal{M}(p_1,\delta'_{12},p_{02})\\ &\cup \bigcup_{\gamma_{01},\delta_{02}} \mathcal{M}(p_{01},\gamma_{01})\times\mathcal{M}_N(\gamma_{01},\delta_{12},\delta_{02})\times\mathcal{M}(\delta_{02},p_{02})\\ &\cup \bigcup_{p'_{02}} \mathcal{M}(p_{01},\delta_{12},p'_{02})\times\mathcal{M}(p'_{02},p_{02})\\ &\cup \bigcup_{\gamma_{02},\delta_{02}} \mathcal{M}(p_{01},\delta_{12},\gamma_{02})\times\mathcal{M}_N(\gamma_{02},\delta_{02})\times\mathcal{M}(\delta_{02},p_{02})\\ &\cup \bigcup_{\gamma_{01},\gamma_{12},\delta_{02}} \mathcal{M}(p_{01},\gamma_{01})\times\mathcal{M}(\delta_{12},\gamma_{12})\times\mathcal{M}_N(\gamma_{01},\gamma_{12},\delta_{02})\times\mathcal{M}(\delta_{02},p_{02}).\end{aligned}$$

(1) $\mathcal{M}(p_{01},\delta_{12},\gamma_{02})$ *is a finite dimensional smooth manifold. If* $\dim\mathcal{M}(p_{01},\delta_{12},\gamma_{02}) = 0$*, then* $\mathcal{M}(p_{01},\delta_{12},\gamma_{02})$ *is compact. If* $\mathcal{M}(p_{01},\delta_{12},\gamma_{02}) = 1$*, then* $\mathcal{M}(p_{01},\delta_{12},\gamma_{02})$

can be compactified into $\overline{\mathcal{M}}(p_{01},\delta_{12},\gamma_{02})$*, whose boundary is*

$$
\begin{aligned}
\partial\overline{\mathcal{M}}(p_{01},\delta_{12},\gamma_{02}) = & \bigcup_{p'_{01}} \mathcal{M}(p_{01},p'_{01}) \times \mathcal{M}(p'_{01},\delta_{12},\gamma_{02}) \\
& \cup \bigcup_{\gamma_{01},\delta_{01}} \mathcal{M}(p_{01},\gamma_{01}) \times \mathcal{M}_N(\gamma_{01},\delta_{01})\mathcal{M}(\delta_{01},\delta_{12},\gamma_{02}) \\
& \cup \bigcup_{p_{12}} \mathcal{M}(\delta_{12},p_{12}) \times \mathcal{M}(p_{01},p_{12},\gamma_{02}) \\
& \cup \bigcup_{\gamma_{12},\delta'_{12}} \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\gamma_{12},\delta'_{12}) \times \mathcal{M}(p_{01},\delta'_{12},\gamma_{02}) \\
& \cup \bigcup_{\delta'_{12}} \mathcal{M}_N(\delta_{12},\delta'_{12}) \times \mathcal{M}(p_{01},\delta'_{12},\gamma_{02}) \\
& \cup \bigcup_{\gamma_{01},\delta_{02}} \mathcal{M}(p_{01},\gamma_{01}) \times \mathcal{M}_N(\gamma_{01},\delta_{12},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}) \\
& \cup \bigcup_{\gamma_{01}} \mathcal{M}(p_{01},\gamma_{01}) \times \mathcal{M}_N(\gamma_{01},\delta_{12},\gamma_{02}) \\
& \cup \bigcup_{p_{02}} \mathcal{M}(p_{01},\delta_{12},p_{02}) \times \mathcal{M}(p_{02},\gamma_{02}) \\
& \cup \bigcup_{\gamma'_{02},\delta_{02}} \mathcal{M}(p_{01},\delta_{12},\gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}) \\
& \cup \bigcup_{\gamma'_{02}} \mathcal{M}(p_{01},\delta_{12},\gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02},\gamma_{02}) \\
& \cup \bigcup_{\gamma_{01},\gamma_{12}} \mathcal{M}(p_{01},\gamma_{01}) \times \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02}) \\
& \cup \bigcup_{\gamma_{01},\gamma_{12},\delta_{02}} \mathcal{M}(p_{01},\gamma_{01}) \times \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\gamma_{01},\gamma_{12},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}).
\end{aligned}
$$

(m) $\mathcal{M}(\delta_{01},\delta_{12},p_{02})$ *is a finite dimensional smooth manifold. If* $\dim\mathcal{M}(\delta_{01},\delta_{12},p_{02}) = 0$*, then* $\mathcal{M}(\delta_{01},\delta_{12},p_{02})$ *is compact. If* $\mathcal{M}(\delta_{01},\delta_{12},p_{02}) = 1$*, then* $\mathcal{M}(\delta_{01},\delta_{12},p_{02})$

can be compactified into $\overline{\mathcal{M}}(\delta_{01}, \delta_{12}, p_{02})$*, whose boundary is*

$$
\begin{aligned}
\partial\overline{\mathcal{M}}(\delta_{01}, \delta_{12}, p_{02}) = & \bigcup_{p_{01}} \mathcal{M}(\delta_{01}, p_{01}) \times \mathcal{M}(p_{01}, \delta_{12}, p_{02}) \\
& \cup \bigcup_{\gamma_{01}, \delta'_{01}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}_N(\gamma_{01}, \delta'_{01}) \times \mathcal{M}(\delta'_{01}, \delta_{12}, p_{02}) \\
& \cup \bigcup_{\delta'_{01}} \mathcal{M}_N(\delta_{01}, \delta'_{01}) \times \mathcal{M}(\delta'_{01}, \delta_{12}, p_{02}) \\
& \cup \bigcup_{\gamma_{12}, \delta_{02}} \mathcal{M}(\delta_{12}, \gamma_{12}) \times \mathcal{M}_N(\delta_{01}, \gamma_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}) \\
& \cup \bigcup_{p_{12}} \mathcal{M}(\delta_{12}, p_{12}) \times \mathcal{M}(\delta_{01}, p_{12}, p_{02}) \\
& \cup \bigcup_{\gamma_{12}, \delta'_{12}} \mathcal{M}(\delta_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{12}, \delta'_{12}) \times \mathcal{M}(\delta_{01}, \delta'_{12}, p_{02}) \\
& \cup \bigcup_{\delta'_{12}} \mathcal{M}_N(\delta_{12}, \delta'_{12}) \times \mathcal{M}(\delta_{01}, \delta'_{12}, p_{02}) \\
& \cup \bigcup_{\gamma_{01}, \delta_{02}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}_N(\gamma_{01}, \delta_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}) \\
& \cup \bigcup_{\delta_{02}} \mathcal{M}_N(\delta_{01}, \delta_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}) \\
& \cup \bigcup_{p'_{02}} \mathcal{M}(\delta_{01}, \delta_{12}, p'_{02}) \times \mathcal{M}(p'_{02}, p_{02}) \\
& \cup \bigcup_{\gamma_{02}, \delta_{02}} \mathcal{M}(\delta_{01}, \delta_{12}, \gamma_{02}) \times \mathcal{M}_N(\gamma_{02}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}) \\
& \cup \bigcup_{\gamma_{01}, \gamma_{12}, \delta_{02}} \mathcal{M}(\delta_{01}, \gamma_{01}) \times \mathcal{M}(\delta_{12}, \gamma_{12}) \times \mathcal{M}_N(\gamma_{01}, \gamma_{12}, \delta_{02}) \times \mathcal{M}(\delta_{02}, p_{02}).
\end{aligned}
$$

(n) $\mathcal{M}(\delta_{01}, \delta_{12}, \gamma_{02})$ *is a finite dimensional smooth manifold. If* $\dim \mathcal{M}(\delta_{01}, \delta_{12}, \gamma_{02}) = 0$*, then* $\mathcal{M}(\delta_{01}, \delta_{12}, \gamma_{02})$ *is compact. If* $\mathcal{M}(\delta_{01}, \delta_{12}, \gamma_{02}) = 1$*, then* $\mathcal{M}(\delta_{01}, \delta_{12}, \gamma_{02})$

can be compactified into $\overline{\mathcal{M}}(\delta_{01},\delta_{12},\gamma_{02})$*, whose boundary is*

$$
\begin{aligned}
\partial\overline{\mathcal{M}}(\delta_{01},\delta_{12},\gamma_{02}) =& \bigcup_{p_{01}} \mathcal{M}(\delta_{01},p_{01}) \times \mathcal{M}(p_{01},\delta_{12},\gamma_{02}) \\
&\cup \bigcup_{\gamma_{01},\delta'_{01}} \mathcal{M}(\delta_{01},\gamma_{01}) \times \mathcal{M}_N(\gamma_{01},\delta'_{01}) \times \mathcal{M}(\delta'_{01},\delta_{12},\gamma_{02}) \\
&\cup \bigcup_{\delta'_{01}} \mathcal{M}_N(\delta_{01},\delta'_{01}) \times \mathcal{M}(\delta'_{01},\delta_{12},\gamma_{02}) \\
&\cup \bigcup_{\gamma_{12},\delta_{02}} \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\delta_{01},\gamma_{12},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}) \\
&\cup \bigcup_{p_{12}} \mathcal{M}(\delta_{12},p_{12}) \times \mathcal{M}(\delta_{01}p_{12},\gamma_{02}) \\
&\cup \bigcup_{\gamma_{12},\delta'_{12}} \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\gamma_{12},\delta'_{12}) \times \mathcal{M}(\delta_{01},\delta'_{12},\gamma_{02}) \\
&\cup \bigcup_{\delta'_{12}} \mathcal{M}_N(\delta_{12},\delta'_{12}) \times \mathcal{M}(\delta_{01},\delta'_{12},\gamma_{02}) \\
&\cup \bigcup_{\gamma_{01},\delta_{02}} \mathcal{M}(\delta_{01},\gamma_{01}) \times \mathcal{M}_N(\gamma_{01},\delta_{12},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}) \\
&\cup \bigcup_{p_{02}} \mathcal{M}(\delta_{01},\delta_{12},p_{02}) \times \mathcal{M}(p_{02},\gamma_{02}) \\
&\cup \bigcup_{\gamma'_{02},\delta_{02}} \mathcal{M}(\delta_{01},\delta_{12},\gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}) \\
&\cup \bigcup_{\gamma'_{02}} \mathcal{M}(\delta_{01},\delta_{12},\gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02},\gamma_{02}) \\
&\cup \bigcup_{\gamma_{01},\gamma_{12}} \mathcal{M}(\delta_{01},\gamma_{01}) \times \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02}) \\
&\cup \bigcup_{\gamma_{01},\gamma_{12},\delta_{02}} \mathcal{M}(\delta_{01},\gamma_{01}) \times \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\gamma_{01},\gamma_{12},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}) \\
&\cup \bigcup_{\gamma_{12}} \mathcal{M}(\delta_{12},\gamma_{12}) \times \mathcal{M}_N(\delta_{01},\gamma_{12},\delta_{02}) \\
&\cup \bigcup_{\gamma_{01}} \mathcal{M}(\delta_{01},\gamma_{01}) \times \mathcal{M}_N(\gamma_{01},\delta_{12},\gamma_{02}) \\
&\cup \bigcup_{\delta_{02}} \mathcal{M}_N(\delta_{01},\delta_{12},\delta_{02}) \times \mathcal{M}(\delta_{02},\gamma_{02}).
\end{aligned}
$$

(o) $\mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02})$ *is a finite dimensional smooth manifold. If* $\dim \mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02}) = 0$*, then* $\mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02})$ *is compact. If* $\mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02}) = 1$*, then* $\mathcal{M}_N(\gamma_{01},\gamma_{12},\gamma_{02})$

can be compactified into $\overline{\mathcal{M}}_N(\gamma_{01}, \gamma_{12}, \gamma_{02})$*, whose boundary is*

$$\begin{aligned}
\partial\overline{\mathcal{M}}_N(\gamma_{01}, \gamma_{12}, \gamma_{02}) = & \bigcup_{\gamma'_{01}} \mathcal{M}_N(\gamma_{01}, \gamma'_{01}) \times \mathcal{M}_N(\gamma'_{01}, \gamma_{12}, \gamma_{02}) \\
& \cup \bigcup_{\gamma'_{12}} \mathcal{M}_N(\gamma_{12}, \gamma'_{12}) \times \mathcal{M}_N(\gamma_{01}, \gamma'_{12}, \gamma_{02}) \\
& \cup \bigcup_{\gamma'_{02}} \mathcal{M}_N(\gamma_{01}, \gamma_{12}, \gamma'_{02}) \times \mathcal{M}_N(\gamma'_{02}, \gamma_{02}).
\end{aligned}$$

Completely, similar arguments hold for $\mathcal{M}_N(\gamma_{01}, \gamma_{12}, \delta_{02}), \mathcal{M}_N(\delta_{01}, \gamma_{12}, \gamma_{02}), \mathcal{M}_N(\delta_{01}, \gamma_{12}, \delta_{02}),$ $\mathcal{M}_N(\gamma_{01}, \delta_{12}, \gamma_{02}), \mathcal{M}_N(\gamma_{01}, \delta_{12}, \delta_{02})$ *and* $\mathcal{M}_N(\delta_{01}, \delta_{12}, \delta_{02})$*. Note that* $\mathcal{M}_N(\delta_{01}, \delta_{12}, \gamma_{02}) = \emptyset$ *because of the maximal principle.*

We omit the proof of Theorem 5.3. We may list every boundary components of 1-dimensional moduli spaces in Theorem 5.3 without omission by chasing intersection points and Reeb chords so that we obtain 1-dimensional moduli spaces after gluing pseudoholomorphic maps. Note that, in Morse homology, there is no broken negative gradient trajectory from a negative boundary critical point to a positive boundary critical point. But, in Floer case, we have broken pseudoholomorphic maps in M which connect a positive Reeb chord and a negative Reeb chord.

We define a linear map $m_2 : C(L_0, L_1) \otimes C(L_1, L_2) \to C(L_0, L_2)$ by

$$
\begin{aligned}
m_2(p_{01}, p_{12}) :=& \sum_{p_{02}} \sharp\mathcal{M}(p_{01}, p_{12}, p_{02}) p_{02} + \sum_{\gamma_{02}} \sharp\mathcal{M}(p_{01}, p_{12}, \gamma_{02})\gamma_{02}, \\
m_2(p_{01}, \gamma_{12}) :=& \sum_{\gamma_{01},\delta_{02},p_{02}} \sharp\mathcal{M}(p_{01}, \gamma_{01})\sharp\mathcal{M}_N(\gamma_{01}, \gamma_{12}, \delta_{02})\sharp\mathcal{M}(\delta_{02}, p_{02}) p_{02} \\
&+ \sum_{\gamma_{01},\delta_{02},\gamma_{02}} \sharp\mathcal{M}(p_{01}, \gamma_{01})\sharp\mathcal{M}_N(\gamma_{01}, \gamma_{12}, \delta_{02})\sharp\mathcal{M}(\delta_{02}, \gamma_{02})\gamma_{02} \\
&+ \sum_{\delta_{12},p_{02}} \sharp\mathcal{M}_N(\gamma_{12}, \delta_{12})\sharp\mathcal{M}(p_{01}, \delta_{12}, p_{02}) p_{02} \\
&+ \sum_{\delta_{12},\gamma_{02}} \sharp\mathcal{M}_N(\gamma_{12}, \delta_{12})\sharp\mathcal{M}(p_{01}, \delta_{12}, \gamma_{02})\gamma_{02} \\
&+ \sum_{\gamma_{01},\gamma_{02}} \sharp\mathcal{M}(p_{01}, \gamma_{01})\sharp\mathcal{M}_N(\gamma_{01}, \gamma_{12}, \gamma_{02})\gamma_{02}, \\
m_2(\gamma_{01}, p_{12}) :=& \sum_{\delta_{01},p_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}(\delta_{01}, p_{12}, p_{02}) p_{02} \\
&+ \sum_{\delta_{01},\gamma_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}(\delta_{01}, p_{12}, \gamma_{02})\gamma_{02}, \\
m_2(\gamma_{01}, \gamma_{12}) :=& \sum_{\delta_{01},\delta_{02},p_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}_N(\delta_{01}, \gamma_{12}, \delta_{02})\sharp\mathcal{M}(\delta_{02}, p_{02}) p_{02} \\
&+ \sum_{\delta_{01},\delta_{02},\gamma_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}_N(\delta_{01}, \gamma_{12}, \delta_{02})\sharp\mathcal{M}(\delta_{02}, \gamma_{02})\gamma_{02} \\
&+ \sum_{\delta_{01},\delta_{12},p_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}_N(\gamma_{12}, \delta_{12})\sharp\mathcal{M}(\delta_{01}, \delta_{12}, p_{02}) p_{02} \\
&+ \sum_{\delta_{01},\delta_{12},\gamma_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}_N(\gamma_{12}, \delta_{12})\sharp\mathcal{M}(\delta_{01}, \delta_{12}, \gamma_{02})\gamma_{02} \\
&+ \sum_{\delta_{01},\gamma'_{01},\delta_{02},p_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}(\delta_{01}, \gamma'_{01})\sharp\mathcal{M}_N(\gamma'_{01}, \gamma_{12}, \delta_{02})\sharp\mathcal{M}(\delta_{02}, p_{02}) p_{02} \\
&+ \sum_{\delta_{01},\gamma'_{01},\delta_{02},\gamma_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}(\delta_{01}, \gamma'_{01})\sharp\mathcal{M}_N(\gamma'_{01}, \gamma_{12}, \delta_{02})\sharp\mathcal{M}(\delta_{02}, \gamma_{02})\gamma_{02} \\
&+ \sum_{\delta_{01},\gamma_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}_N(\delta_{01}, \gamma_{12}, \gamma_{02})\gamma_{02} \\
&+ \sum_{\delta_{01},\gamma'_{01},\gamma_{02}} \sharp\mathcal{M}_N(\gamma_{01}, \delta_{01})\sharp\mathcal{M}(\delta_{01}, \gamma'_{01})\sharp\mathcal{M}_N(\gamma'_{01}, \gamma_{12}, \gamma_{02})\gamma_{02},
\end{aligned}
$$

where the dimension of each moduli space is 0.

Note that the definition of m_2 is more complicated than the cup product in Morse complex. But, as in the Morse case, Theorem 4.8, we can prove the following theorem by observing the boundary of 1-dimensional moduli spaces of pseudoholomorphic maps in Theorem 5.3. We omit the proof.

Theorem 5.4. *We denote by* $\partial_{01} : C(L_0, L_1) \to C(L_0, L_1), \partial_{12} : C(L_1, L_2) \to C(L_1, L_2)$ *and* $\partial_{02} : C(L_0, L_2) \to C(L_0, L_2)$ *the boundary operators of Floer complexes. Then we obtain the Leibniz rule:*

$$\partial_{02} m_2(*_{01}, *_{12}) = m_2(\partial_{01} *_{01}, *_{12}) \pm m_2(*_{01}, \partial_{12} *_{12}),$$

where $*_{ij}$ *is a generator of* $C(L_i, L_j)$.

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