J-holomorphic curves in symplectic topology

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1 Symplectic manifolds

$\mathbb{R}^{2n} = \{(x_1, y_1, \ldots, x_n, y_n)\}$

Definition $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$

Hamiltonian differential equations

$$\frac{dc_{2i-1}}{dt}(t) = \frac{\partial H}{\partial y_i}(t, c_1(t), \ldots, c_{2n}(t))$$

$$\frac{dc_{2i}}{dt}(t) = -\frac{\partial H}{\partial x_i}(t, c_1(t), \ldots, c_{2n}(t))$$

Definition $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R} \quad H_t(\cdot) := H(t, \cdot)$

Hamiltonian vector field on $\mathbb{R}^{2n}$

$$X_{H_t} := \left(\frac{\partial H_t}{\partial y_1}, -\frac{\partial H_t}{\partial x_1}, \ldots, \frac{\partial H_t}{\partial y_n}, -\frac{\partial H_t}{\partial x_n} \right)$$

Remark $c : \mathbb{R} \to \mathbb{R}^{2n}$

Hamiltonian differential equation

$$\Leftrightarrow \frac{dc}{dt}(t) = X_{H_t}(c(t))$$

Definition $\psi : \mathbb{R}^{2n} = \{(x_1, y_1, \ldots, x_n, y_n)\} \to \mathbb{R}^{2n} = \{(X_1, Y_1, \ldots, X_n, Y_n)\}$ diffeomorphism

ψ: canonical transformation

$\overset{def}{\Longleftrightarrow} \psi^* \sum_{i=1}^{n} dX_i \wedge dY_i = \sum_{i=1}^{n} dx_i \wedge dy_i$

Lemma $\tilde{H} := H \circ \psi^{-1} \quad \tilde{c} := \psi \circ c$

$$\frac{dc}{dt}(t) = X_{H_t}(c(t)) \Leftrightarrow \frac{d\tilde{c}}{dt}(t) = X_{\tilde{H}_t}(\tilde{c}(t))$$

Definition M : smooth manifold

ω : 2-form on M

ω : symplectic form

$\overset{def}{\Longleftrightarrow}$ • ω is non-degenerate i.e.
$\omega(u, v) = 0$ for $\forall v \Rightarrow u = 0$
• ω is closed i.e. $d\omega = 0$

Definition (M, ω) : symplectic manifold

Example $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$

Theorem (Darboux)
(M, ω) : symplectic manifold

$\Rightarrow \forall p \in M \ \exists U$: open neighborhood of p
$\exists \varphi : U \to \varphi(U) \subset \mathbb{R}^{2n}$: diffeomorphism
s.t. $\omega = \varphi^* \sum_{i=1}^{n} dx_i \wedge dy_i$ on U

Remark Symplectic structue is locally trivial

Lemma (M, ω) : symplectic manifold
$H : \mathbb{R} \times M \to \mathbb{R}$: smooth function
$\Rightarrow \exists! X_{H_t}$: vector field on M s.t.

$$dH_t = \omega(X_{H_t}, \cdot)$$

Remark From now on we consider periodic H
i.e. $H(t+1, \cdot) = H(t, \cdot)$ and closed orbits
$c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ i.e.

$$\frac{dc}{dt}(t) = X_{H_t}(c(t))$$

Theorem (Gromov '85)

(M, ω) : closed symplectic and $\pi_2(M) = 0$

$\Rightarrow \exists c : S^1 \to M$ s.t. $\dfrac{dc}{dt}(t) = X_{H_t}(c(t))$

Theorem (Floer '89)

(M, ω) : closed symplectic and $\pi_2(M) = 0$

Suppose all solutions of $\dfrac{dc}{dt}(t) = X_{H_t}(c(t))$ are non-degenerate.

$$\Rightarrow \sharp \text{ of solutions } c \geq \sum_{i=0}^{2n} \dim H_i(M; \mathbb{Z}_2)$$

2 J-holomorphic curves

Definition M : smooth manifold

$J = \{J(p)\}_{p \in M}$: almost complex structure
$\overset{def}{\Longleftrightarrow}$ $J(p) : T_pM \to T_pM$ linear map
s.t. $J(p)^2 = -\mathrm{id}_{T_pM}$

(M, J) : almost complex manifold

Example Complex manifolds

Definition (M, J) : almost complex manifold
(Σ, j) : Riemann surface
$u : \Sigma \to M$: smooth map

u : J-holomoprhic (or pseudoholomorphic)
$\overset{def}{\Longleftrightarrow} J \circ du = du \circ j$

Remark $du = u_* : T_z\Sigma \to T_{u(z)}M$

Remark If J is integrable, J-holomorphic is holomorphic.

Definition Cauchy–Riemann operator

$$\bar{\partial}_J u := \frac{1}{2}(du + J(u) \circ du \circ j)$$

Definition Cauchy–Riemann equation

$$\bar{\partial}_J u = 0$$

Remark u : J-holomorphic $\Leftrightarrow \bar{\partial}_J u = 0$

Remark $\Sigma \supset U = \{s + \sqrt{-1}t\}$ complex coordinate

$$\bar{\partial}_J u = 0 \Leftrightarrow \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0$$

Definition (M, ω) : symplectic manifold
J : almost complex structure on M

J : ω-compatible

$$\overset{def}{\Longleftrightarrow} \bullet\ v \neq 0 \Rightarrow \omega(v, Jv) > 0$$
$$\bullet\ \omega(Jv, Jw) = \omega(v, w)$$

Theorem ω-compatible J exist

Remark J : ω-compatible
$\Rightarrow g(v, w) := \omega(v, Jw)$ is a Riemannian metric on M s.t. $g(v, w) = g(Jv, Jw)$

Remark (Σ, j) : Riemann surface

μ : non-vanishing 2-form on Σ

$\Rightarrow h(\xi, \zeta) := \mu(\xi, j\zeta)$ is a Riemannian metric on Σ s.t. $h(\xi, \zeta) = h(j\xi, j\zeta)$

Remark g and h induce an inner product on $T^*\Sigma \otimes u^*TM$

Remark $du \in \Gamma(T^*\Sigma \otimes u^*TM)$

Definition $u : (\Sigma, j) \to (M, J)$: smooth map

Energy of u

$$E(u) := \frac{1}{2}\int_\Sigma |du|^2 \mu$$

Lemma $E(u)$ does not depend on μ.

Lemma $E(u) = \int_\Sigma |\bar{\partial}_J u|^2 \mu + \int_\Sigma u^* \omega$

Corollary u : J-holomorphic

$\Rightarrow E(u) = \int_\Sigma u^* \omega \ (\geq 0)$ topological invariant

Corollary u : J-holomorphic

$\Rightarrow E(u) = \int_\Sigma u^*\omega \ (\geq 0)$ topological invariant

Corollary u : J-holomorphic and $\int_\Sigma u^*\omega = 0$

$\Rightarrow u$ is a constant map.

3 Basic properties of J-holomorphic curves

- Mean value inequality
- Minimal energy
- Removal of singularities
- Convergence I
- Finiteness of singularities
- Convergence II
- Bubbling phenomenon

Definition $D_r := \{z \in \mathbb{C} \mid |z| \leq r\}$

Theorem (Mean value inequality)

(M, J) : closed almost complex manifold

$\Rightarrow \exists \hbar > 0$ and $\exists C > 0$ s.t. if $u : D_r \to M$ is J-holomorphic s.t. $E(u) < \hbar$, then

$$|du(0)|^2 \leq \frac{C}{r^2} \int_{|z| \leq r} |du|^2 dxdy$$

Theorem (Minimal energy)
(M, J) : closed almost complex manifold
(Σ, j) : closed Riemann surface

$\Rightarrow \exists \hbar > 0$ s.t. $\forall u : \Sigma \to M$: non-constant
J-holomorphic $\Rightarrow E(u) \geq \hbar$

Proof In the case of $\Sigma = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$
Suppose $E(u) < \hbar \Rightarrow$

$$|du(z_0)|^2 \leq \frac{C}{r^2} \int_{|z-z_0|<r} |du|^2 dxdy$$
$$\leq C\hbar/r^2 \to 0 \ (r \to \infty) \qquad \square$$

Theorem (Removal of singularities)

(M, ω) : closed symplectic manifold

$u : D_r \setminus \{0\} \to M$: J-holomorphic

Suppose $E(u) < \infty$

$\Rightarrow \exists \bar{u} : D_r \to M$: J-holomorphic

s.t. $\bar{u}|_{D_r \setminus \{0\}} = u$

Theorem (Convergence I)

(M, J) : closed almost complex manifold

(Σ, j) : Riemann surface

$u_i : \Sigma \to M,\ i = 1, 2, \ldots$: J-holomorphic

Suppose $\sup_i \sup_{z \in \Sigma} |du_i(z)| < \infty$.

$\Rightarrow \exists u_{i_j} : \Sigma \to M$: subsequence and

$\exists u : \Sigma \to M$: J-holomorphic

s.t. $u_{i_j} \to u$ in C^∞-topology on $\forall$ compact subset of Σ

Definition $u_i : \Sigma \to M$, $i = 1, 2, \ldots$:
J-holomorphic

$z_\infty \in \Sigma$: singular point

$\overset{def}{\Longleftrightarrow} \exists z_i \in \Sigma$, $i = 1, 2, \ldots$

s.t. $\lim_{i \to \infty} z_i = z_\infty$ and $\lim_{i \to \infty} |du_i(z_i)| = \infty$

Definition $z_\infty \in \Sigma$: singular point

$\overset{def}{\Longleftrightarrow} \exists z_i \in \Sigma$, $i = 1, 2, \ldots$

s.t. $\lim_{i \to \infty} z_i = z_\infty$ and $\lim_{i \to \infty} |du_i(z_i)| = \infty$

Theorem (Finiteness of singularities)

(M, J) : closed almost complex manifold

(Σ, j) : Riemann surface

$u_i : \Sigma \to M$, $i = 1, 2, \ldots$: J-holomorphic

Suppose $\sup_i E(u_i) < \infty$

$\Rightarrow \sharp$ of singular points is finite.

Theorem (Convergence II)

(M, J) : closed almost complex manifold

$u_i : \Sigma \to M$, $i = 1, 2, \ldots$: J-holomorphic

Suppose $\sup_i E(u_i) \leq C$

$\Rightarrow \exists\{\zeta_1, \ldots, \zeta_k\} \subset \Sigma \ \ \exists u_{i_j}$: subsequence

$\exists u : \Sigma \setminus \{\zeta_1, \ldots, \zeta_k\} \to M$: J-holomorphic

s.t. $u_{i_j} \to u$ in C^∞-topology on $\forall$ compact subset of $\Sigma \setminus \{\zeta_1, \ldots, \zeta_k\}$ and $E(u) \leq C$

Remark $\exists \bar{u} : \Sigma \to M$: J-holomorphic s.t. $\bar{u}|_{\Sigma \setminus \{\zeta_1, \ldots, \zeta_k\}} = u$ (Removal of singularities)

Theorem (Bubbling phenomenon)

(M, J) : closed almost complex manifold

$u_i : \Sigma \to M$, $i = 1, 2, \dots$: J-holomorphic

Suppose $\sup_i E(u_i) \leq C$

$z_i \in \Sigma$: $\lim_{i\to\infty} z_i = z_\infty$ and $\lim_{i\to\infty} |du_i(z_i)| = \infty$

Define $c_i := |du_i(z_i)|$ and $v_i(z) := u_i\left(z_i + \frac{z}{c_i}\right)$

where $z \in \mathbb{C}$ is a complex coordinate around z_i

Remark • $|dv_i(0)| = 1$

• $E(v_i) \leq C$

Define $c_i := |du_i(z_i)|$ and $v_i(z) := u_i\left(z_i + \frac{z}{c_i}\right)$ where $z \in \mathbb{C}$ is a complex coordinate around z_i

$\Rightarrow \exists\{\zeta_1, \ldots, \zeta_k\} \subset \mathbb{C}\ \ \exists v_{i_j}$: subsequence
$\exists v : \mathbb{C} \setminus \{\zeta_1, \ldots, \zeta_k\} \to M$: non-constant J-holomorphic
s.t. $v_{i_j} \to v$ in C^∞-topology on $\forall$ compact subset of $\mathbb{C} \setminus \{\zeta_1, \ldots, \zeta_k\}$ and $E(v) \leq C$

Remark $\exists \bar{v} : \mathbb{C}P^1 \to M$: J-holomorphic s.t. $\bar{v}|_{\mathbb{C}P^1 \setminus \{\zeta_1, \ldots, \zeta_k, \infty\}} = v$ (Removal of singularities)

4 Existence of periodic orbits

Theorem (Gromov '85)

(M,ω) : closed symplectic and $\pi_2(M)=0$

$H : S^1 \times M \to \mathbb{R}$: smooth function

$\Rightarrow \exists c : S^1 \to M$ s.t. $\dfrac{dc}{dt}(t) = X_{H_t}(c(t))$

Definition $\rho : [0, \infty) \times \mathbb{R} \to [0, 1]$: smooth s.t.

- $\operatorname{supp} \rho(r, \cdot) \subset [-r, r]$
- $\partial\rho/\partial s \geq 0$ for $s \leq 0$
- $\partial\rho/\partial s \leq 0$ for $0 \leq s$
- when $1 \leq r$, $\rho(r, s) = 1$ for $s \in [-r + 1, r - 1]$

Remark $\rho(0, s) = 0$

Definition $u : \mathbb{C}P^1 \to M$: smooth map

perturbed Cauchy–Riemann operator

$$\bar{\partial}_{J,H,r} u := \frac{\partial u}{\partial s}(s,t) + J(u(s,t))\frac{\partial u}{\partial t}(s,t) \\ -\rho(r,s)J(u(s,t))X_{H_t}(u(s,t))$$

where we identify $\mathbb{C}P^1 \setminus \{0,\infty\} \cong \mathbb{R} \times S^1$

Definition perturbed Cauchy–Riemann equation

$$\bar{\partial}_{J,H,r} u = 0$$

Remark $\bar{\partial}_{J,H,0} = \bar{\partial}_J$

Definition $u : \mathbb{C}P^1 \to M$: smooth map

$$E_{\mathbb{C}P^1}(u) := \int_{\mathbb{R}\times S^1} \left| \frac{\partial u}{\partial s} \right|^2 dsdt$$

where we identify $\mathbb{C}P^1 \setminus \{0, \infty\} \cong \mathbb{R} \times S^1$

Lemma $\bar{\partial}_{J,H,r} u = 0$

$$\Rightarrow E_{\mathbb{C}P^1}(u) \leq \int_{\mathbb{C}P^1} u^* \omega$$
$$+ \int_{S^1} \left(\max_{x\in M} H_t(x) - \min_{x\in M} H_t(x) \right) dt$$

Theorem 1 (M, ω) : closed symplectic manifold and $\pi_2(M) = 0$

$\Rightarrow \forall r \geq 0 \ \exists u$ s.t. $\bar{\partial}_{J,H,r} u = 0$

Proof Suppose $\exists r_0 \geq 0$ s.t. no solution of $\bar{\partial}_{J,H,r_0} u = 0$.

Define the moduli space of the solutions of perturbed Cauchy–Riemann equation

$$\mathcal{M} := \left\{ (r, u) \;\middle|\; \begin{array}{l} u_*[\mathbb{C}P^1] = 0 \in H_2(M; \mathbb{Z}) \\ r \in [0, r_0] \;\; \bar{\partial}_{J,H,r} u = 0 \end{array} \right\}$$

Step 1 $\mathcal{M}$ is a $(2n+1)$-dim "smooth manifold" with boundary.

Remark $\{(r_0, u) \in \mathcal{M}\} = \emptyset$ by assumption

Remark $\partial\mathcal{M} = \{(0, u) \in \mathcal{M}\}$

Remark $(0, u) \in \partial\mathcal{M} = \{(0, u) \in \mathcal{M}\}$
$\Rightarrow u$ is J-holomorphic and constant
$\Rightarrow \partial\mathcal{M} \cong M$

Step 2 $\mathcal{M}$ is compact.

Proof Take a sequence $u_i \in \mathcal{M}$, $i = 1, 2, \dots$
Suppose $\sup_i \sup_{z \in \mathbb{C}P^1} |du_i(z)| = \infty$.
Since $\sup_i E_{\mathbb{C}P^1}(u_i) < \infty$

$\Rightarrow \exists$ non-constant J-holomorphic curve $v : \mathbb{C}P^1 \to M$ (Bubbling phenomenon). But it is impossible since $\pi_2(M) = 0$.
$\Rightarrow \sup_i \sup_{z \in \mathbb{C}P^1} |du_i(z)| < \infty$.
$\Rightarrow \exists u_{i_j}$: convergent subsequence

Definition evaluation map $ev : \mathcal{M} \to M$

$$ev\big((r, u)\big) := u(0)$$

Remark $0 \in \mathbb{C} \subset \mathbb{C}P^1$

So far we obtain

- $(2n+1)$-dim compact smooth manifold $\mathcal{M}$
- $ev : \mathcal{M} \to M$
- $ev(\partial\mathcal{M}) = M$

$\Rightarrow [M] = 0 \in H_{2n}(M; \mathbb{Z})$ Contradiction!

$\Rightarrow \forall r \geq 0 \ \exists u$ s.t. $\bar{\partial}_{J,H,r} u = 0$ $\square$

Theorem 2 (M,ω) : closed symplectic manifold and $\pi_2(M)=0$
(r_i,u_i), $i=1,2,\ldots$: $r_i\nearrow\infty$ and $\bar\partial_{J,H,r_i}u_i=0$
Suppose $\sup_i E_{\mathbb{C}P^1}(u_i)<\infty$

$\Rightarrow \exists u_{i_j}$: subsequence and $\exists u:\mathbb{R}\times S^1\to M$: solution of

$$\frac{\partial u}{\partial s}+J(u)\left(\frac{\partial u}{\partial t}-X_{H_t}(u)\right)=0$$

s.t. $u_{i_j}\to u$ in C^∞-topology on $\forall$ compact subset of $\mathbb{R}\times S^1$ and $E_{\mathbb{C}P^1}(u)<\infty$

Proof Suppose $\sup_i \sup_{z\in\mathbb{C}P^1} |du_i(z)| = \infty$.
Since $\sup_i E_{\mathbb{C}P^1}(u_i) < \infty$
$\Rightarrow \exists$ non-constant J-holomorphic curve $v : \mathbb{C}P^1 \to M$ (Bubbling phenomenon). But it is impossible since $\pi_2(M) = 0$.
$\Rightarrow \sup_i \sup_{z\in\mathbb{C}P^1} |du_i(z)| < \infty$.
$\Rightarrow \exists u_{i_j}$: convergent subsequence and the limit $u : \mathbb{R} \times S^1 \to M$ satisfies

$$\frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_{H_t}(u)\right) = 0$$

□

Theorem 3 (M, ω) : closed symplectic manifold
$u : \mathbb{R} \times S^1 \to M$: solution of

$$\frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_{H_t}(u)\right) = 0$$

Suppose $E_{\mathbb{C}P^1}(u) < \infty$

$\Rightarrow \exists c : S^1 \to M$ s.t. $\dfrac{dc}{dt}(t) = X_{H_t}(c(t))$

Proof Recall $g(Jv, Jw) = g(v, w)$ and

$$E_{\mathbb{C}P^1}(u) = \int_{\mathbb{R} \times S^1} \left|\frac{\partial u}{\partial s}\right|^2 ds dt$$

$$\Rightarrow \int_{\mathbb{R}\times S^1} \left| \frac{\partial u}{\partial t} - X_{H_t}(u) \right|^2 ds dt < \infty$$

$\Rightarrow \exists s_i \nearrow \infty$ s.t.

$$\int_{S^1} \left| \frac{\partial u}{\partial t}(s_i, t) - X_{H_t}(u(s_i, t)) \right|^2 dt \to 0$$

Put $c(t) := \lim_{i\to\infty} u(s_i, t)$

$$\Rightarrow \frac{dc}{dt}(t) = X_{H_t}(c(t)) \qquad \square$$

Theorem 1, 2 & 3 imply the Gromov's theorem.

Thank you!