On the Coordinate Ring of $SL(3,\mathbb{C})$-Character Varieties under Poisson Reduction

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Overview

Representation varieties and Character varieties

Representation varieties

- π, a finitely generated and finitely presented group.
- G, connected reductive algebraic group (reductive: maximal connected nilpotent subgroup is trivial).

Then the representation variety is defined to be

$$R(\pi, G) := Hom_{gp}(\pi, G).$$

Topologies on representation varieties

- When π is a r-rank free group F_r. There is a one-to-one correspondence.

$$R(F_r, G) \leftrightarrow G^r.$$

- For general π. Take a free group representation $F_r \twoheadrightarrow \pi$. There is a one-to-one correspondence.

$$R(\pi, G) \hookrightarrow R(F_r, G) \leftrightarrow G^r.$$

This defines the topology on the representation varieties $R(F_r, G)$ and $R(\pi, G)$.

Action on representation varieties

There is a natural action, G-conjugation, on $R(\pi, G)$.

$$\begin{aligned} G \times R(\pi, G) &\to R(\pi, G) \\ (g, \rho) &\mapsto g\rho g^{-1}. \end{aligned}$$

Character variety

- $\mathfrak{X}(\pi, G) := R(\pi, G)/G$.
- For G is a complex algebraic group. The G-character variety is defined by the GIT-quotient

$$\mathfrak{X}(\pi, G) := R(\pi, G)//G := specm\mathbb{C}[R(\pi, G)]^G.$$

  Hence, we have $\mathbb{C}[\mathfrak{X}(\pi, G)] = \mathbb{C}[R(\pi, G)]^G$.
- When $\pi = F_r$, define the rank r G-character variety $\mathfrak{X}_r(G)$ to be the $\mathfrak{X}(F_r, G)$.

Trace coordinate

Example

Let $G = SL_m(\mathbb{C})$ and $m \geq 2$. Using the Newton's identities, we have

$$\mathbb{C}[\mathfrak{X}_1(G)] = \mathbb{C}[R(F_1, G)]^G \simeq \mathbb{C}[G]^G = \mathbb{C}[tr_1, tr_2, \cdots, tr_{m-1}].$$

So

$$\mathfrak{X}_1(G) = \mathbb{C}^{m-1}.$$

Here, tr_x is defined as follows. Let $x \in \pi$ ($\pi := F_1 = \mathbb{Z}$ in the Example). Define the trace function with respect to x by

$$\begin{aligned} tr_x : \mathfrak{X}(\pi, G) &\to \mathbb{C} \\ [\rho] &\mapsto tr_x(\rho) := tr(\rho(x)) \end{aligned}$$

Trace coordinate

Theorem

Let $G = SL_m(\mathbb{C})$. Then $\mathbb{C}[\mathfrak{X}_n(G)]$ is finitely generated by trace functions.

Trace coordinate

Example

Let $G = SL_2(\mathbb{C})$ and $F_2 =< x, y >$. Then

$$\mathbb{C}[\mathfrak{X}_2(G)] = \mathbb{C}[G^2]^G = \mathbb{C}[tr_x, tr_y, tr_{xy}] = \mathbb{C}[\mathbb{C}^3].$$

In other words, the isomorphism is

$$\begin{aligned} \mathfrak{X}_2(G) &\simeq \mathbb{C}^3 \\ [(X, Y)] &\mapsto (tr(X), tr(Y), tr(XY)), \end{aligned}$$

where $X = \rho(x)$ and $Y = \rho(y)$ for $\rho \in R(F_2, G)$.

Various Quotients

Definition

$\mathfrak{X}^{T_2}(\pi, G) :=$ Hausdorffication$(R(\pi, G)/G)$

If G is complex, then $\mathfrak{X}^{T_2} = \mathfrak{X}^{GIT}$.

Various Quotients

Example

$$R(\pi, SU_n)/SU_n = \mathfrak{X}^{T_2}(\pi, SU_n).$$

Example

$$R(F_1, SL_2(\mathbb{R}))/SL_2(\mathbb{R}) = \mathbb{R} \cup \{\pm 2', \pm 2''\}$$

with $\overline{\{\pm 2'\}} = \pm\{2, 2'\}$ and $\overline{\{\pm 2''\}} = \pm\{2, 2''\}$.
So

$$\mathfrak{X}_1^{T_2}(SL_2(\mathbb{R})) = \mathbb{R}.$$

Embedding

Theorem (Carlos Florentino and Sean Lawton)

Let G be a compact connected group and $G_{\mathbb{C}}$ be its complexification. Then there is a natural embedding

$$\mathfrak{X}_r(G) \hookrightarrow \mathfrak{X}_r(G_{\mathbb{C}}),$$

which is also a strong deformation retraction via the Kempf-Ness set.

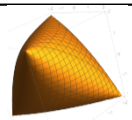
$$\mathbb{C}[\mathfrak{X}_r(G)] := Restriction\left(\mathbb{C}[\mathfrak{X}_r(G_{\mathbb{C}})]\right).$$

SL/SU Character variety

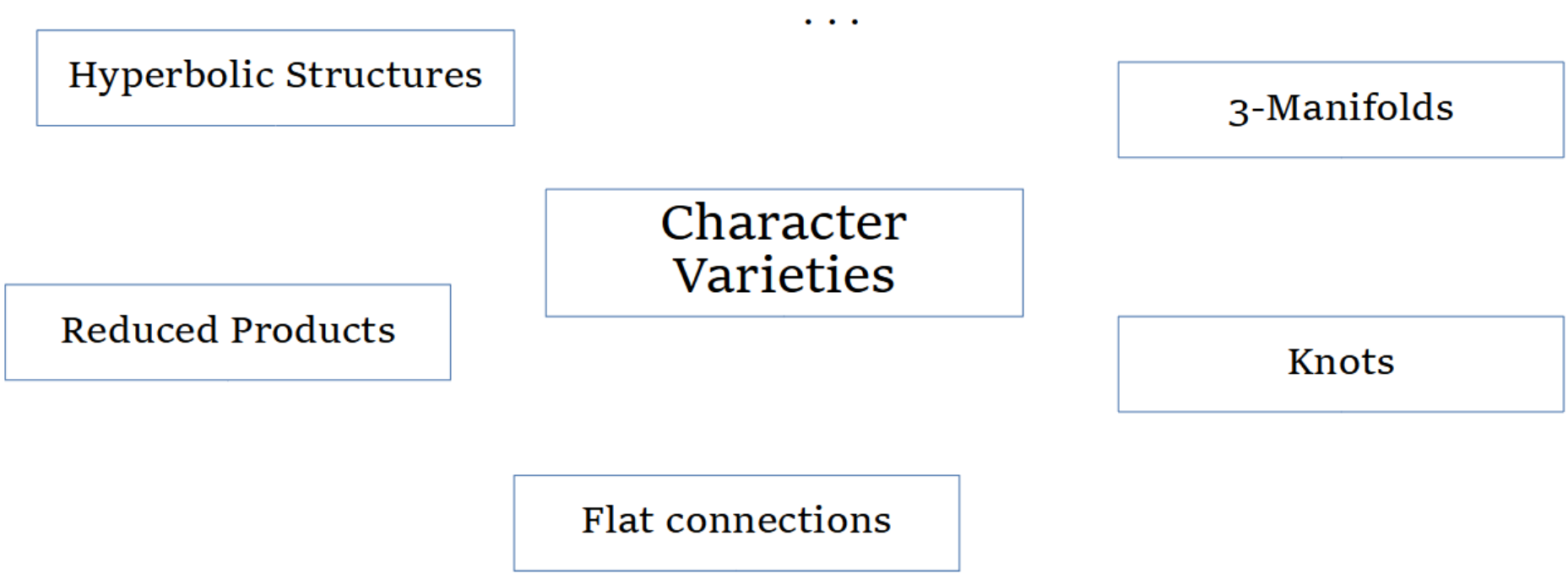
The table of $\mathfrak{X}_n(SL_m(\mathbb{C}))$.

m \ n	1	2	3	⋯	n
2	$\mathbb{C}^1$	$\mathbb{C}^3$	$M^6 \subseteq \mathbb{C}^7$		
3	$\mathbb{C}^2$	$M^8 \subseteq \mathbb{C}^9$			
⋮				$\ddots$	
m	$\mathbb{C}^{m-1}$				$M^{(m^2-1)(n-1)}$ for $n > 1$

The table of $\mathfrak{X}_n(SU_m)$.

m \ n	1	2	⋯	n
2	$[2, -2]$		S^6	
3		S^8		
⋮			$\ddots$	
m				$M^{(m^2-1)(n-1)}$ for $n > 1$

Applications for Character Varieties



Surface group

Let $\pi = \pi_1(\Sigma_{g,n})$ and $G = SL_m(\mathbb{C})$ with $m > 1$.
Note that for $n \geq 1$, π is free of rank $2g + n - 1$.

Theorem (Lawton)

For $n, n' \geq 1$. $\mathfrak{X}(\Sigma_{g,n}, G)$ and $\mathfrak{X}(\Sigma_{g',n'}, G)$ are the same as varieties if and only if $\chi(\Sigma_{g,n}) = \chi(\Sigma_{g',n'})$.

Poisson Structure

Poisson Structure

Theorem (Goldman)

$\mathfrak{X}(\Sigma_{g,n}, G)$ has a Poisson structure.

Theorem (Goldman)

For $n = 0$. $\mathfrak{X}(\Sigma_g, G)$ has a symplectic structure.

Theorem (Goldman)

For $n, n' \geq 1$. $\mathfrak{X}(\Sigma_{g,n}, G)$ and $\mathfrak{X}(\Sigma_{g',n'}, G)$ are the same as Poisson varieties if and only if $(g, n) = (g', n')$.

Poisson Formula

For $[\alpha], [\beta] \in \pi_1(\Sigma_{g,n})$, $tr_\alpha, tr_\beta \in \mathbb{C}[\mathfrak{X}(\Sigma_{g,n}, G)]$. Then Poisson braket of two functions can be calculated by this formula

$$\{tr_\alpha, tr_\beta\} = \sum_{p \in \alpha \cap \beta} sign(p, \alpha, \beta) \left(tr_{\alpha_p \beta_p} - \frac{1}{m} tr_\alpha tr_\beta \right),$$

where $\alpha_p \beta_p$ means the way α and β concatenate in $\pi_1(\Sigma_{g,n}, p)$.

Remark

$$\{tr_\alpha, tr_\beta\} = 0$$

whenever α and β are disjoint.

Symplectic leaves

When $n \geq 1$, we define the boundary map by

$$b : \mathfrak{X}_{2g+n-1}(G) = \mathfrak{X}(\Sigma_{g,n}, G) \to \prod_{i=1}^{n} \mathfrak{X}(\partial^{(i)}\Sigma_{g,n}, G) \simeq (\mathbb{C}^{m-1})^n$$
$$[\rho] \mapsto \left([\rho|_{\pi_1(\partial^{(i)}\Sigma_{g,n}, G)}]\right)_i.$$

Theorem

The filber of the boundary map b are symplectic leaves.

We call those filber by relative character varieties.

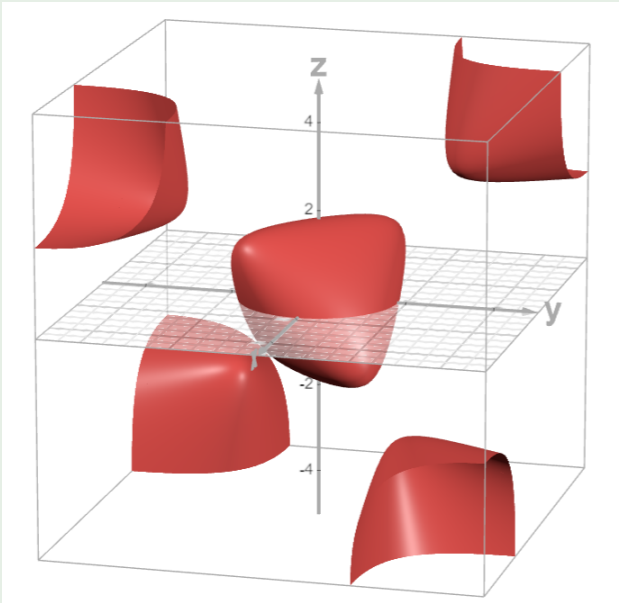
Symplectic leaves

Example

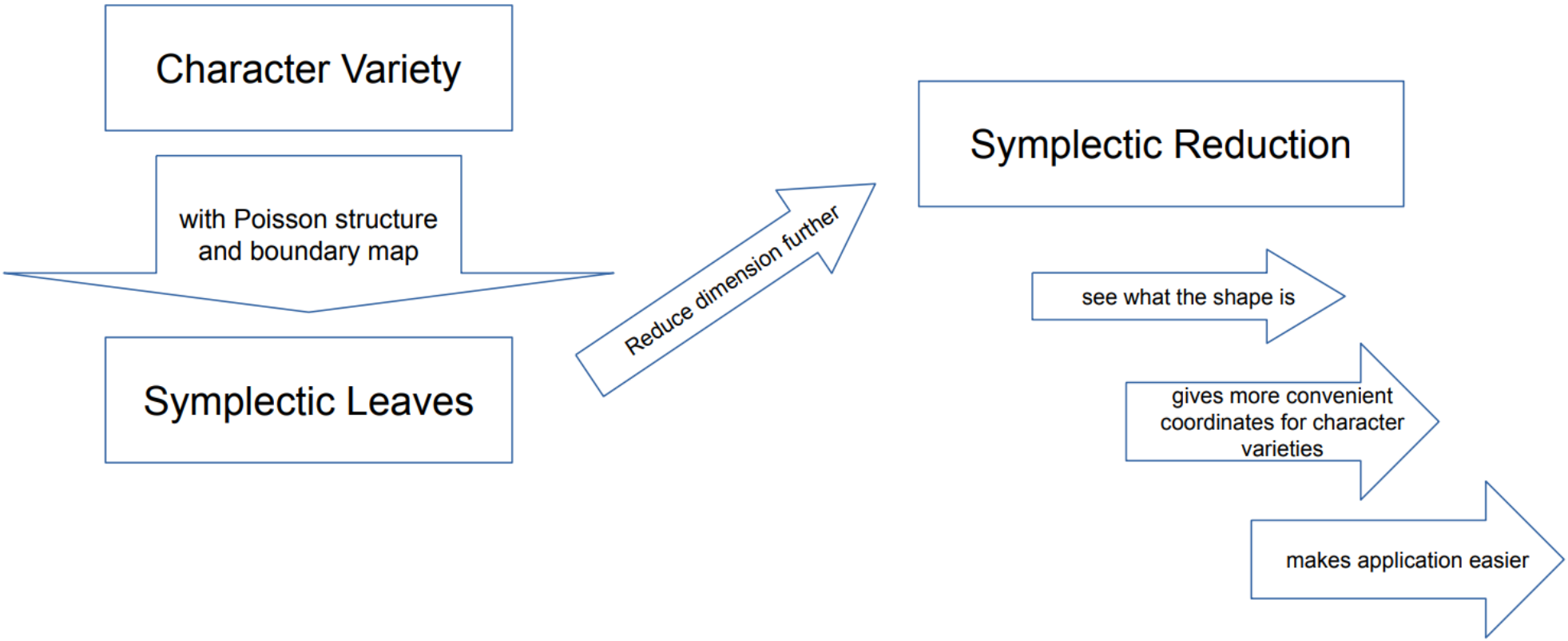
$$b: \quad \mathfrak{X}(\Sigma_{1,1}, SL_2(\mathbb{C})) \to \mathbb{C}$$

$$b: \quad \mathfrak{X}(\Sigma_{1,1}, SU_2) \to [-2, 2]$$

$$[(X, Y)] \mapsto tr(XYX^{-1}Y^{-1}).$$



Character Varieties



Symplectic leaves

Example

Consider the case for $\Sigma_{1,1}$ and $G = SL_3(\mathbb{C})$.

$$b : \mathfrak{X}(\Sigma_{1,1}, G) = \mathfrak{X}_2(G) \to \mathbb{C}^2$$
$$[(X, Y)] \mapsto (t_5, t_{-5}),$$

where $t_5 = tr(XYX^{-1}Y^{-1})$ and $t_{-5} = tr(YXYX^{-1}Y^{-1})$.

Let M_b be a relative character varieties in this example. The dimension of M_b are 4 for generic cases.

Symplectic Reduction

Define a moment map h from M_b to $\mathbb{C}^2$ by

$$[(X, Y)] \to (t_2, t_{-2}) := (tr(Y), tr(Y^{-1})).$$

We have a complex 2-torus, $T^2_{\mathbb{C}} = (\mathbb{C}^\times)^2$, actions on each filber $M_{b,h}$ of the moment map.

$$T^2_{\mathbb{C}} \times M_{b,h} \to M_{b,h}$$
$$(D, [(X, Y)]) \mapsto [(XD, Y)]$$

where D is a 3 by 3 diagonal matrix with determinate 1.

Symplectic Reduction

Define $S_{b,h} := M_{b,h}/T^2_{\mathbb{C}}$ to be the symplectic reduction for each value h, which is only 2-dim in generic cases.

Theorem (_and Eugene Xia)

$$\mathbb{C}[S_{b,h}] = \frac{\mathbb{C}[z, z^*, \vec{w}]}{(t_5 + t_{-5} - P_{zw}, t_5 t_{-5} - Q_{zw}, zz^* - w_1 w_2 w_3)},$$

where P_{zw} and Q_{zw} are polynomials in z, z^* and $\vec{w}$, but its too long to put them here. Moreover, the Poisson brackets of those generators can be calculated.

Symplectic Reduction

$$
\begin{aligned}
P_{zw} = {} & (-a_1^2a_2 - a_1^2a_3 + a_1a_2^2 + a_1a_3^2 - a_2^2a_3 - a_2a_3^2 + 2)\,w_1 \\
& + (a_1^2a_2 - a_1^2a_3 - a_1a_2^2 - a_1a_3^2 - a_2^2a_3 + a_2a_3^2 + 2)\,w_2 \\
& + (-a_1^2a_2 + a_1^2a_3 - a_1a_2^2 - a_1a_3^2 + a_2^2a_3 - a_2a_3^2 + 2)\,w_3 \\
& + a_1^2a_2 + a_1^2a_3 + a_1a_2^2 + a_1a_3^2 + a_2^2a_3 + a_2a_3^2
\end{aligned}
$$

$$
\begin{aligned}
Q_{zw} = {} & (-a_1^4a_2^2 - a_1^4a_3^2 + 2a_1^3a_2^3 + 2a_1^3a_3^3 + 2a_1^3 - a_1^2a_2^4 - 2a_1^2a_2 - a_1^2a_3^4 - 2a_1^2a_3 - 2a_1a_2^2 - 2a_1a_3^2 \\
& \quad - a_2^4a_3^2 + 2a_2^3a_3^3 + 2a_2^3 - a_2^2a_3^4 - 2a_2^2a_3 - 2a_2a_3^2 + 2a_3^3 + 6)\,z \\
& + (-a_1^4a_2^2 - a_1^4a_3^2 + 2a_1^3a_2^3 + 2a_1^3a_3^3 + 2a_1^3 - a_1^2a_2^4 - 2a_1^2a_2 - a_1^2a_3^4 - 2a_1^2a_3 - 2a_1a_2^2 - 2a_1a_3^2 \\
& \quad - a_2^4a_3^2 + 2a_2^3a_3^3 + 2a_2^3 - a_2^2a_3^4 - 2a_2^2a_3 - 2a_2a_3^2 + 2a_3^3 + 6)\,z^* \\
& + (a_1^3 - 2a_1^2a_2 - 2a_1^2a_3 + a_1a_2^2 + a_1a_3^2 + a_2^3a_3^3 - 2a_2^2a_3 - 2a_2a_3^2 + 4)\,w_1^2 \\
& + (a_1^4a_3^2 - a_1^3a_3^3 - a_1^3 + a_1^2a_2 + a_1a_2^2 + a_1a_3^2 + a_2^4a_3^2 - a_2^3a_3^3 - a_2^3 + a_2a_3^2 - 2)\,w_1w_2 \\
& + (a_1^4a_2^2 - a_1^3a_2^3 - a_1^3 + a_1^2a_3 + a_1a_2^2 + a_1a_3^2 - a_2^3a_3^3 + a_2^2a_3^4 + a_2^2a_3 - a_3^3 - 2)\,w_1w_3 \\
& + (-a_1^3a_2^3 - a_1^3a_3^3 - 2a_1^3 + a_1^2a_2^4 + 2a_1^2a_2 + a_1^2a_3^4 + 2a_1^2a_3 + a_1a_2^2 + a_1a_3^2 - 2a_2^3a_3^3 - a_2^3 + 2a_2^2a_3 + 2a_2a_3^2 - a_3^3 - 4)\,w_1 \\
& + (a_1^3a_3^3 + a_1^2a_2 - 2a_1^2a_3 - 2a_1a_2^2 - 2a_1a_3^2 + a_2^3 - 2a_2^2a_3 + a_2a_3^2 + 4)\,w_2^2 \\
& + (-a_1^3a_2^3 - a_1^3a_3^3 + a_1^2a_2^4 + a_1^2a_2 + a_1^2a_3^4 + a_1^2a_3 - a_2^3 + a_2^2a_3 + a_2a_3^2 - a_3^3 - 2)\,w_2w_3 \\
& + (a_1^4a_2^2 - a_1^3a_2^3 - 2a_1^3a_3^3 - a_1^3 + a_1^2a_2 + 2a_1^2a_3 + 2a_1a_2^2 + 2a_1a_3^2 - a_2^3a_3^3 - 2a_2^3 + a_2^2a_3^4 + 2a_2^2a_3 + a_2a_3^2 - a_3^3 - 4)\,w_2 \\
& + (a_1^3a_2^3 - 2a_1^2a_2 + a_1^2a_3 - 2a_1a_2^2 - 2a_1a_3^2 + a_2^2a_3 - 2a_2a_3^2 + a_3^3 + 4)\,w_3^2 \\
& + (a_1^4a_3^2 - 2a_1^3a_2^3 - a_1^3a_3^3 - a_1^3 + 2a_1^2a_2 + a_1^2a_3 + 2a_1a_2^2 + 2a_1a_3^2 + a_2^4a_3^2 - a_2^3a_3^3 - a_2^3 + a_2^2a_3 + 2a_2a_3^2 - 2a_3^3 - 4)\,w_3 \\
& + a_1^3a_2^3 + a_1^3a_3^3 + a_1^3 + a_2^3a_3^3 + a_2^3 + a_3^3 + 3
\end{aligned}
$$

where a_1, a_2 and a_3 are roots of $\alpha^3 - tr(X)\alpha^2 + tr(X^{-1})\alpha - 1$.

Poisson Reduction

Let $S_{G,h}$ be the gluing of all $S_{b,h}$ where values b runs through some specific Zariski open set.

We define $S_{K,h}$ for $K = SU_3$ in the similar way.

Theorem (_and Eugene Xia)

There exists a strong deformation retraction on $S_{G,h}$ that retracts to $S_{K,h}$ for each value h.

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Thank you for your listening