Homological mirror symmetry for weighted projective spaces and Morse homotopy

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- Homological mirror symmetry (HMS) asserts that for a Calabi-Yau manifold M and its mirror manifold $\check{M}$, there exists an equivalence

$$Tr(Fuk(M)) \simeq D^b(coh(\check{M})).$$

- Since then, beyond the the original setting of mirror pairs of Calabi-Yau manifolds, HMS has been studied for a lager class of mirror pairs with some adjustments in setting.
- Today, we would like to talk about a version of HMS for weighted projective spaces based on the Strominger-Yau-Zaslow's (SYZ) point of view.

The outline of this talk

1. Introduction
2. orbifolds and weighted projective spaces
3. SYZ construction and SYZ transformation
    - toric orbifolds
    - SYZ construction
    - SYZ transformation
4. HMS for WPS
    - Mo(P)
    - $DG(\mathbb{P}(q_0, \ldots, q_n))$
    - main theorem

Background : SYZ

Strominger-Yau-Zaslow's point of view.

- Strominger-Yau-Zaslow proposed a geometric interpretation of Mirror symmetry, in particular, a construction of mirror pairs as torus fibrations which are fiberwise dual to each other over the same base. They also proposed a version of Fourier-Mukai transform along torus fibers. These are the so-called SYZ construction and the SYZ transformation.

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & \check{X} \arrow[dl, "\check{\pi}"] \\ & B & \end{tikzcd}$$

Background : HMS from the point of view of SYZ

- Kontsevich and Soibelman ('01) discussed homological mirror symmetry (HMS) from the point of view of SYZ.

$$\begin{array}{ccc} T^*B/\Lambda^* & & TB/\Lambda \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\check{\pi}} & \\ & B & \end{array}$$

- They based on Fukaya-Oh's result ([Fukaya-Oh '97]), which gives an equivalence $Fuk(T^*B) \simeq Mo(B)$. They introduced a weighted version of the category of Morse homotopy $Mo(B)$ and considered $Fuk(T^*B/\Lambda^*) \simeq Mo(B)$ for the torus fibration π.

- Futaki-Kajiura discussed HMS in this line when a complex manifold is a smooth compact toric manifold by modifying the category of weighted Morse homotopy to the case where the base space has boundaries.
- Let X be a smooth Kähler toric manifold with a moment map $\mu : X \to P$. Then, if we write the restriction $\check{\pi} := \mu|_{\check{Y}}$ to the open dense torus orbit $\check{Y} \simeq (\mathbb{C}^*)^{\dim X}$, this $\check{\pi}$ gives a torus fibration on the interior of the moment polytope P. Applying the SYZ construction to $\check{\pi}$ we obtain its dual torus fibration π.

$$\begin{array}{ccc} Y & & \check{Y} \subsetneq X \\ & {}^{\pi}\searrow \quad \swarrow{}^{\mu|_{\check{Y}} = \check{\pi}} & \\ & \mathrm{Int}P & \end{array}$$

- Then, they introduced the category $Mo(P)$ of the moment polytope P, and proposed HMS as an equivalence of triangulated categories of the form

$$Tr(Mo_{\mathcal{E}}(P)) \simeq D^b(coh(X)),$$

where $\mathcal{E}$ is an ordered finite set of Lagrangian sections which are SYZ mirror to a full exceptional collection of $D^b(coh(X))$ (if there exists).

- In this formulation, HMS for some smooth toric varieties have been shown:
  - $X = \mathbf{P}^n \times \mathbf{P}^m$ (by Futaki-Kajiura '21),
  - $X = \mathbf{F}_1$ (by Futaki-Kajiura '22),
  - $X =$the remaining two cases of toric Fano surfaces, i.e., $\mathrm{Bl}_{2\mathrm{pts}}(\mathbf{P}^2), \mathrm{Bl}_{3\mathrm{pts}}(\mathbf{P}^2)$ (by Nakanishi '23),
  - $X = \mathbf{F}_k, k \geq 1$ (by Nakanishi '24).

The aim of this talk is to give a toric orbifold analogue of this formulation of HMS, and show HMS for weighted projective spaces $\mathbb{P}(q_0, \ldots, q_n)$ with $\gcd(q_0, \ldots, q_n) = 1$.

Orbifolds

- An orbifold is defined as a pair $\mathcal{X} = (X, \mathcal{U})$ of a (Hausdorff) topological space X with an orbifold atlas $\mathcal{U} = \{(\widetilde{U}_i, \Gamma_i, \psi_i)\}_i$.
- orbifold chart: For an open set $U \subset X$, an orbifold chart is a triple $(\widetilde{U}, \Gamma, \psi)$, where $\widetilde{U} \simeq \mathbb{C}^n$ is a complex Euclidean space, Γ is a finite group which acts on $\widetilde{U}$ effectively, and $\psi : \widetilde{U} \to U$ is a surjective map which induces a homeomorphism $U \simeq \widetilde{U}/\Gamma$.

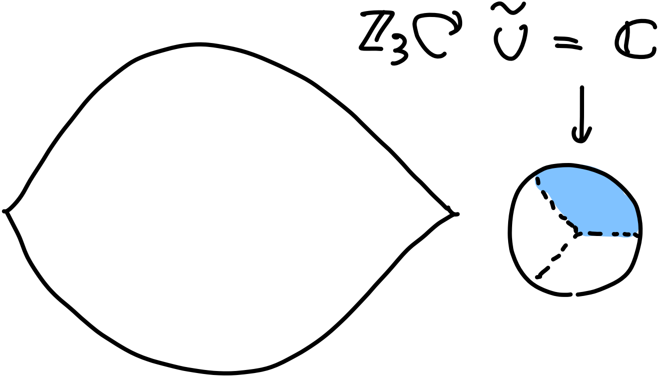
Weighted projective spaces (WPS)

- Let $q_0, \ldots, q_n$ be positive integers with $\gcd(q_0, \ldots, q_n) = 1$. Consider $\mathbb{C}^*$ action on $\mathbb{C}^{n+1} \backslash \{0\}$ given by

$$\lambda \cdot (z_0, \ldots, z_n) = (\lambda^{q_0} z_0, \ldots, \lambda^{q_n} z_n).$$

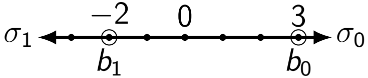
  We denote by $\mathbf{P}(q_0, \ldots, q_n)$ the quotient space.
- The weighted projective space (of type $Q = (q_0, \ldots, q_n)$) is an orbifold $\mathbb{P}(Q) = (\mathbf{P}(Q), \mathcal{U})$.
- An orbifold chart for $U_i := \{z_i = 1\}/\mathbb{C}^*$ is $(\widetilde{U}_i, \Gamma_i, \psi_i)$, where $\tilde{U}_i = \mathbb{C}^n \simeq \{z_i = 1\} \subset \mathbb{C}^{n+1} \backslash \{0\}$, $\Gamma_i = \{\text{the } q_i\text{-th roots of unity}\} \simeq \mathbb{Z}/q_i\mathbb{Z}$, and $\psi_i : \tilde{U}_i \to U_i =$ the quotient map.

the weighted projective line $\mathbb{P}(3, 2)$



toric orbifolds

- $N \simeq \mathbb{Z}^n$: lattice, $M = \mathrm{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$: dual lattice
- $\Sigma = \{\sigma_0, \sigma_1, \dots\} \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$: a complete and simplicial fan
- $\{b_i\}$: stacky vectors. For each ray ρ_i, we take a lattice point $b_i \in N \cap \rho_i$.
- $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$ gives rise to a toric orbifold $\mathcal{X}_{\mathbf{\Sigma}}$ (= a toric DM stack with trivial generic stabilizers in the sense of [Borisov-Chen-Smith '08]).
- Example. The stacky fan of $\mathbb{P}(3, 2)$ is as follows.



- $\sigma \in \Sigma(n)$ endows orbifold charts : Let $N_\sigma = \bigoplus_{\rho_i \subset \sigma} \mathbb{Z} b_i$.
  $(\widetilde{U}_\sigma = \mathrm{Spec}(\mathbb{C}[M_\sigma \cap \sigma^\vee]) \simeq \mathbb{C}^n, N/N_\sigma,$ the quotient map)
- Example ($\mathbb{P}(3, 2)$). $N/N_{\sigma_0} = \mathbb{Z}/3\mathbb{Z}, N/N_{\sigma_1} = \mathbb{Z}/2\mathbb{Z}$

Next we would like to consider the SYZ construction as an analogue of that in the case of smooth toric: (Here, we follow the notation from [Chan '09].) a smooth toric variety has open dense torus $(\mathbb{C}^*)^n = N \otimes \mathbb{C}^*$, and it can be expressed as

$$TN_{\mathbb{R}}/N = N_{\mathbb{R}} \times N_{\mathbb{R}}/N$$

and its SYZ mirror is obtained as

$$T^*N_{\mathbb{R}}/M = N_{\mathbb{R}} \times M_{\mathbb{R}}/M.$$

A toric orbifold has the action of DM torus, which has the open dense torus orbit isomorphic to the algebraic torus $(\mathbb{C}^*)^n = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ ([Iwanari '09, Fantechi-Mann-Nironi '10]). We apply the above description to the orbifold cases.

SYZ construction

- Consider an affine manifold $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ with an open covering $\{(N_\sigma)_{\mathbb{R}}\}_{\sigma \in \Sigma(n)}$, where $(N_\sigma)_{\mathbb{R}} = \oplus_{\rho_i \subset \sigma} \mathbb{R} b_i$.
- Let $Y := T^* N_{\mathbb{R}} / 2\pi M \to N_{\mathbb{R}}$, $\check{Y} := T N_{\mathbb{R}} / 2\pi N \to N_{\mathbb{R}}$.

$$Y|_{(N_{\sigma_i})_{\mathbb{R}}} \simeq (N_{\sigma_i})_{\mathbb{R}} \times 2\pi (M_{\sigma_i})_{\mathbb{R}} / 2\pi M$$

$$\check{Y}|_{(N_{\sigma_i})_{\mathbb{R}}} \simeq (N_{\sigma_i})_{\mathbb{R}} \times 2\pi (N_{\sigma_i})_{\mathbb{R}} / 2\pi N$$

- When σ is not smooth, i.e., $N_\sigma \subsetneq N$ (and $M \subsetneq M_\sigma$), the periods of torus fibers are affected by these differences.
- Example ($\mathbb{P}(3,2)$). For $p \in (N_{\sigma_0})_{\mathbb{R}}$,

$$Y_p \simeq \mathbb{R}/(2\pi \cdot 3\mathbb{Z}),\ \check{Y}_p \simeq \mathbb{R} \Big/ \left(2\pi \cdot \frac{1}{3}\mathbb{Z}\right).$$

SYZ constructions by using moment maps

In the smooth case, if we equip X with a Kähler structure and equip $\check{Y}$ with the induced Kähler structure, then we have the Legendre transformation $N_{\mathbb{R}} \xrightarrow{\simeq} \mathrm{Int}\,P, \check{x}_k \mapsto \frac{\partial \phi}{\partial \check{x}_k} =: x^k$, where $\mathrm{Int}\,P$ denotes the interior of a moment polytope. Then, we can treat dual torus fibrations as that over the base $B := \mathrm{Int}\,P$.

$$
\begin{tikzcd}
Y \arrow[dr] & & \check{Y} \arrow[dl] \\
& \mathrm{Int}\,P &
\end{tikzcd}
$$

In the case of WPS, we practically equip $\check{Y}$ with a Kähler structure (given by a two-form on $\mathbb{P}(Q)$, possibly degenerate along toric divisors). We denote the image by B and its closure by $P = \overline{B}$. We can identify B with $N_{\mathbb{R}}$ via the Legendre transformation as in the smooth case, and we treat $Y, \check{Y}$ as the SYZ fibrations over B as well.

SYZ transformation

- Let us breifly recall that the SYZ transformation due to [Leung-Yau-Zaslow '00].
- Let $\underline{s}$ be a section of Y. A section $\underline{s}$ is Lagrangian section if and only if a lift s of it to $T^*N_{\mathbb{R}}$ is a closed one form, hence locally exact.
- A lift of a section $s = \{s^\sigma = (s^{\sigma 1}, \dots, s^{\sigma n})\}_{\sigma \in \Sigma(n)}$ gives a connection of the form

$$D_{s,\sigma} := d - \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{n} s^{\sigma l}(\check{x}) d\check{y}_{\sigma k} \tag{3.1}$$

on the trivial line bundle over the mirror manifold $\check{Y}$.
- s is Lagrangian iff D_s is holomorphic, i.e., $D_s^{0,2} = 0$.

how to construct Lagrangian sections corr. holo. line bdl's on $\mathcal{X}$?

- In order to discuss HMS, we focus on Lagrangian sections such that the corresponding line bundles with connections on $\check{Y}$ can be extended to over the whole $\mathcal{X}$.
- In terms of SYZ point of view, an equivalence of HMS should be interpreted as the SYZ transformaition. In the HMS set-up in [Futaki-Kajiura '21], the symplectic counterpart of $D^b(coh(X))$ consists of such Lagrangian sections.
- They started by $(\mathcal{L}, D)$ over X, and reconstruct a Lagrangian section by comparing the expression above and $D|_{\check{Y}}$ with some twist to cancel $d\check{x}_i$-terms.
- Let us demonstrate this in the weighted projective spaces, as toric orbifolds.

construction of Lagrangian sections corresp. $\mathcal{O}(a)$

Let $\mathcal{O}(a) = \mathcal{O}(\sum_{i=0}^{n} a_i \mathcal{D}_i)$ be an (orbifold) line bundle on $\mathbb{P}(q_0, \ldots, q_n)$, where a_i's are integers with $\sum_{i=0}^{n} q_i a_i = a$. We equip it with a connection D_a. If we take a mermorphic section η of $\mathcal{O}(a)$, the restriction $\eta|_{\check{Y}}$ is nowhere vanishing and holomoprhic, and hence gives a frame of the trivial line bundle $\mathcal{O}(a)|_{\check{Y}}$. With respect to this frame, we can equip it with the following connection

$$D_a|_{\check{Y}} + \sum_k a_{\sigma_i k}(d\check{x}_{\sigma_i k} + \sqrt{-1} d\check{y}_{\sigma_i k}),$$

where $(a_{\sigma_i 1}, \ldots, a_{\sigma_i n}) \in M_{\sigma_i}$ for $\sigma_i \in \Sigma(n)$, which is given by the local expression of $\eta = \{\eta_{\sigma_i} = \prod w_{\sigma_i k}^{a_{\sigma_i k}}\}$. We twist it with Ψ_a to cancel the $d\check{x}_{\sigma_i k}$-terms.

$$\Psi_a^{-1} \left(D_a|_{\check{Y}} + \sum_k a_{\sigma_i k}(d\check{x}_{\sigma_i k} + \sqrt{-1} d\check{y}_{\sigma_i k}) \right) \Psi_a.$$

By comparing this and the expression seen before, we obtain a Lagrangian section s_a of $T^*N_{\mathbb{R}}$.

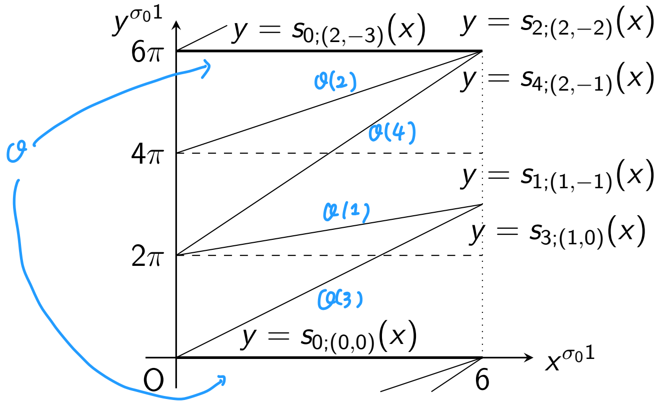
$$\begin{pmatrix} y^{\sigma_i 1} \\ \vdots \\ y^{\sigma_i n} \end{pmatrix} = \begin{pmatrix} s^{\sigma_i 1}_{a;K_a}(x) \\ \vdots \\ s^{\sigma_i n}_{a;K_a}(x) \end{pmatrix} = 2\pi \frac{a}{2q_0 \dots q_n} \begin{pmatrix} x^{\sigma_i 1} \\ \vdots \\ x^{\sigma_i n} \end{pmatrix} - 2\pi \begin{pmatrix} a_0 \\ \vdots \\ a_{i-1} \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix},$$

where $K_a = (a_0, \dots, a_n) \in \mathbb{Z}^{\Sigma(1)}$ satisfying $\sum_{i=0}^n q_i a_i = a$. It descends to a Lagrangian section $\underline{s_a}$ of $Y = T^*N_{\mathbb{R}}/M$. It is well-defined, even though the shift part of s_a lies in M_σ ($M \subset M_\sigma \subset M \otimes_{\mathbb{Z}} \mathbb{Q}$) and it depends on the choice of a meromorphic section.

Example. For $\mathcal{O}(a)$ on $\mathbb{P}(3,2)$, the corresponding Lagrangian section has a lift $s_{a;K_a}$ locally of the form

$$s^{\sigma_0 1}_{a;(a_0,a_1)} = 2\pi\frac{a}{2}\cdot\frac{x^{\sigma_0 1}}{6} - 2\pi a_1$$

for some $K_a = (a_0, a_1) \in \mathbb{Z}^2$ satisfying $3a_0 + 2a_1 = a$. In the figure below, note that $M = 3\mathbb{Z} \subset M_{\sigma_0} = \mathbb{Z} \subset M \otimes_{\mathbb{Z}} \mathbb{Q}$.



Summary so far and Towards HMS

- So far, we consider the SYZ picture for the open dense torus $\check{Y}$ of $\mathcal{X}_{\boldsymbol{\Sigma}} = \mathbb{P}(q_0, \ldots, q_n)$ and its dual torus fibration $\pi : Y \to B$ over the interior $B = \mathrm{Int}P$ of the polytope.

$$\begin{array}{ccc} Y & & \check{Y} \subsetneq \mathcal{X}_{\boldsymbol{\Sigma}} \\ & {}^{\pi}\searrow \quad \swarrow^{\check{\pi}} & \\ & \mathrm{Int}\,P & \end{array}$$

- For HMS, what we should consider is the whole $\mathcal{X}_{\boldsymbol{\Sigma}}$ and $D^b(coh(\mathcal{X}))$ on the complex side. Note that ∂P corresponds to toric divisors, which we removed from $\mathcal{X}_{\boldsymbol{\Sigma}}$ to obtain $\check{Y} \simeq \mathcal{X}_{\boldsymbol{\Sigma}} \setminus \bigcup_{\rho_i \in \Sigma(1)} \mathcal{D}_i$.
- On the symplectic side, [Futaki-Kajiura '21] discussed the category $Mo(P)$ of weighted Morse homotopy on the moment polytope, where they based on [Fukaya-Oh '97, Kontsevich-Soibelman '01].

The main theorem is as follows.

Theorem (N. '25)

There exists a linear A_∞-equivalence

$$Mo_{\mathcal{E}}(P) \simeq DG_{\mathcal{E}}(\mathbb{P}(q_0, \dots, q_n)).$$

- $DG(\mathcal{X})$ is a dg category consisiting of line bundles equipped with connections.
- For $\mathcal{X} = \mathbb{P}(q_0, \dots, q_n)$, let $\mathcal{E} = (\mathcal{O}, \dots, \mathcal{O}(\sum_{j=0}^{n} q_j - 1))$. It is known that $\mathcal{E}$ forms a full strongly exceptional colelction of $D^b(coh(\mathbb{P}(q_0, \dots, q_n))) \simeq Tr(DG_{\mathcal{E}}(\mathbb{P}(q_0, \dots, q_n)))$.
- The cateogry $Mo(P)$ of weihgted Morse homotopy on the polytope consists of Lagrangian setions.
- $Mo_{\mathcal{E}}(P)$ is the full subcategory consisting of Lagrangian sections of $\mathcal{E}$, (which is SYZ mirror to the above $\mathcal{E}$).

Category of weighted Morse homotopy on P

The category of weighted Morse homotpy on P is defined as follows ([Futaki-Kajiura '21]).

- Objects of $Mo(P)$ are graphs of Lagrangian sections of $\pi : Y(= T^*N_{\mathbb{R}}/2\pi M) \to B$ such that the SYZ transform of it ($=$ a trivial linebundle with a holomorphic connection over $\check{Y}$) can be extended to the whole $\mathcal{X}$.
- We assume that these Lagrangian sections L, L' intersect cleanly in the following sence. There exists an open set $\tilde{B} \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ such that $\tilde{B}$ includes $P = \overline{B}$ and L, L' can be smoothly extended over $\tilde{B}$ so that they intersect cleanly over $\tilde{B} \supset P$.

- For (L, L'), the hom space $Mo(P)(L, L')$ is set to be the $\mathbb{Z}$-graded vector space (over $\mathbb{C}$)

$$Mo(P)(L, L') = \bigoplus_{V \subset \pi(L \cap L')} \mathbb{C}[V],$$

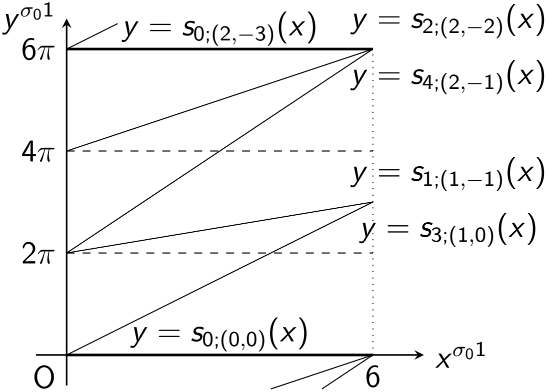
where V is connected component of $\pi(L \cap L') \subset P$ satisfying a certain condition.

- The $\mathbb{Z}$-grading $|V|$ of V is given as follows. Note that for each L we can locally take smooth function $f_L : \tilde{B} \to \mathbb{R}$ so that a lift $\overline{L} = \mathrm{graph}(df_L)$. Let S_v be the stable manifold of $-\mathrm{grad}(f_L - f_{L'})$ with $v \in V$. $|V| = \dim S_v$.
- Remark. Here, we consider the Morse cohomoly degree.
- Remark. $Mo^0(P)(L, L) = \mathbb{C}[P]$.

- A_∞ product is given by counting gradient trees up to smooth homotopy with an appropriate weight.
- Rather than going into full details, in order to comupte the full subcategory $Mo_{\mathcal{E}}(P)$ we only need to compute $\mathfrak{m}_2 =$ the composition of morphism. (We will explain it later.)

Example of computations of the hom space

In order to calculate $Mo_{\mathcal{E}}(P)(L_a, L_b)$, take lifts s_a, s_b of Lagrangian sections to the covering space $\overline{T^*N_{\mathbb{R}}}$ over P and consider the nonempty intersections of the graphs.



For $b = 4$, $a = 1$, $Mo(P)(L_a, L_b) = \mathbb{C}[0]$, $0 \in \partial[0, 6]$.

More generally, connected components of $\pi(L_a \cap L_b) \subset P$ can be calculated by solving $s_{b;K_b}(x) - s_{a;K_a}(x) = 0, x \in P$. The solution in the case of $a \neq b$ is locally expressed as

$$x^{\sigma_i} = \frac{2q_0 \cdots q_n}{b-a}(K_b^{\sigma_i} - K_a^{\sigma_i}).$$

We denote it by $v_{ab;K_{ab}}$ with the index $K_{ab} := K_b - K_a$, namely, $K_{ab} = (k_0, \ldots, k_n) \in \mathbb{Z}^{n+1}$ with

$$\sum_{i=0}^{n} q_i k_i = b - a.$$

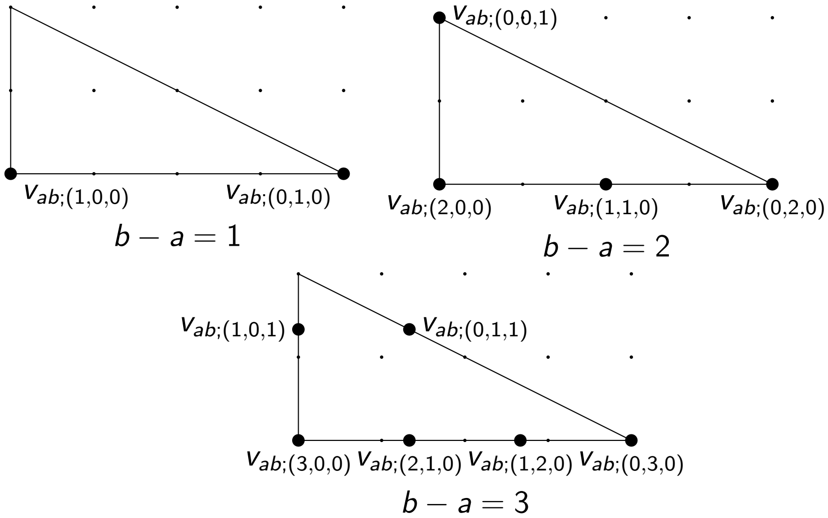
We denote the connected component by $V_{ab;K_{ab}} = \{v_{ab;Ka_ab}\}$. Note that $V_{ab;K_{ab}} \subset P$ iff $K_{ab} \in (\mathbb{Z}_{\geq 0})^{n+1}$. We have

$$Mo^r_{\mathcal{E}}(P)(L_a, L_b) = \begin{cases} \mathbb{C}[P] \quad \text{for } a = b,\ r = 0, \\ \bigoplus_{\substack{K_{ab}=(k_0,\ldots,k_n)\in\mathbb{Z}^{n+1}_{\geq 0} \\ \sum_{j=0}^{n} q_j k_j = b-a}} \mathbb{C}[V_{ab;K_{ab}}] \text{ for } a < b,\ r = 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

Remarks

- The all generators of $Mo^0_{\mathcal{E}}(L_a, L_b)$ belong to the boundary ∂P except for P in $Mo^0_{\mathcal{E}}(L_a, L_a)$.
- The nontrivial parts of the hom space $Mo_{\mathcal{E}}(P)(L_a, L_b)$ appear only in degree zero and for $a \leq b$. So, for the degree reason, $Mo_{\mathcal{E}}(P)$ is a DG category, i.e., $\mathfrak{m}_k = 0, k \geq 3$. We have the trivial differential, $\mathfrak{m}_1 = 0$, and we see that $\mathcal{E}$ forms a full strongly exceptional collection in $Tr(Mo_{\mathcal{E}}(P))$.
- It remains to compute $\mathfrak{m}_2 =$ composition of morphisms.

Example ($\mathbb{P}(1,1,2)$). The generators of $Mo^0_{\mathcal{E}}(P)(L_a, L_b)$, $0 < b - a \leq 3$, are illustrated in the following figure, where the smaller dots are the lattice points in $M_{\sigma_0} = \mathbb{Z}^2$.



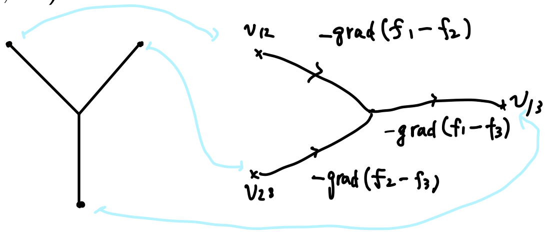
gradient trees

We next recall gradient trees in order to define $\mathfrak{m}_2$.

Fer a triple (L_1, L_2, L_3), take generators $V_{12} \in Mo(P)(L_1, L_2)$ and $V_{23} \in Mo(P)(L_2, L_3)$. We locally take smooth functions $f_1, f_2, f_3 : \tilde{B} \to \mathbb{R}$ so that $\underline{L_i} = \text{graph}(df_i)$ and $V_{12} \subset \pi(\underline{L_1} \cap \underline{L_2})$ and $V_{23} \subset \pi(\underline{L_2} \cap \underline{L_3})$. Let V_{13} be a connected component of $\pi(\underline{L_1} \cap \underline{L_3})$.

For points $v_{12} \in V_{12}, v_{23} \in V_{23}, v_{13} \in V_{13}$, a gradient tree starting at v_{12}, v_{23} and ending at v_{13} is a continuous map $\gamma : T \to P$ with a rooted trivalent tree T satisfying the following condition.

- Regarding as a planar tree, the left (resp. right) leaf vertex maps to v_{12} (resp. v_{13}), and the root vertex maps to v_{13}.
- For each edge e including the leaf vertex, we have $\frac{d\gamma|_e}{dt} = -\mathrm{grad}(f_i - f_{i+1}), e \simeq (-\infty, 0]$. For the edge e including the root vertex, we have $\frac{d\gamma|_e}{dt} = -\mathrm{grad}(f_1 - f_3)$, $e \simeq [0, \infty)$.



We denote the set of gradient trees starting at v_{12}, v_{23} and ending at v_{13} by $\mathcal{GT}(v_{12}, v_{23}; v_{13})$.

We denote the set of gradient trees by

$$\mathcal{GT}(V_{12}, V_{23}; V_{13}) := \bigcup_{(v_{12}, v_{23}; v_{13}) \in V_{12} \times V_{23} \times V_{13}} \mathcal{GT}(v_{12}, v_{23}; v_{13})$$

and write

$$\mathcal{HGT}(V_{12}, V_{23}; V_{13}) := \mathcal{GT}(V_{12}, V_{23}; V_{13})/\text{smooth homotopy}.$$

Then, we define the composition $\mathfrak{m}_2$ by

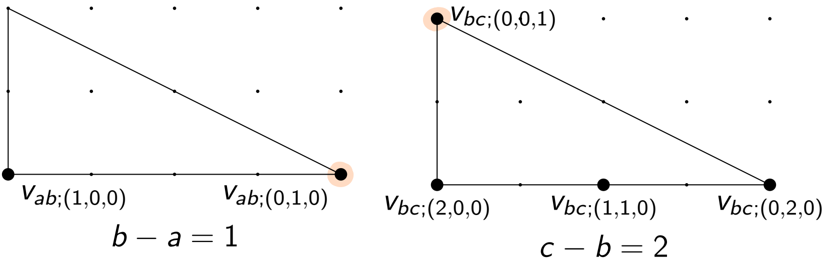
$$\mathfrak{m}_2 : Mo(P)(L_1, L_2) \otimes Mo(P)(L_2, L_3) \to Mo(P)(L_1, L_3),$$
$$\mathfrak{m}_2(V_{12}, V_{23}) = \sum_{|V_{13}| = |V_{12}| + |V_{23}|} \sum_{[\gamma] \in \mathcal{HGT}(V_{12}, V_{23}; V_{13})} e^{-A(\gamma)} V_{13}$$

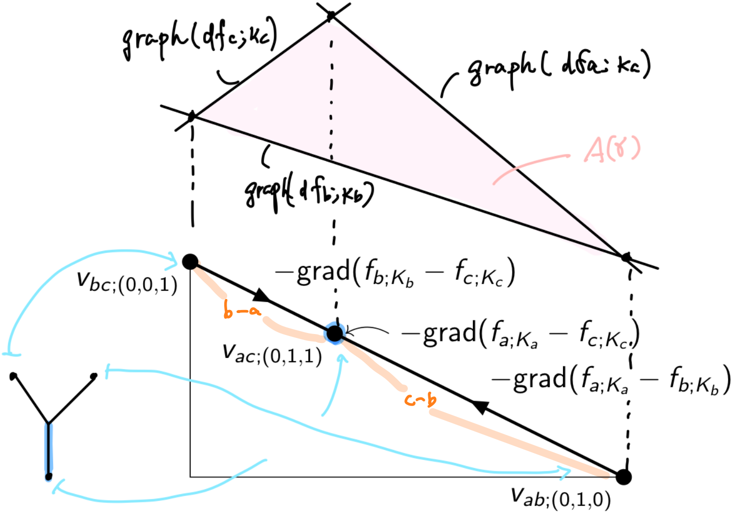
where $|V_{13}| = |V_{12}| + |V_{23}|$ and $A(\gamma)$ denotes the symplectic area of disk in $\pi^{-1}(\gamma(T))$ as in [Kontsevich-Soibelman].

Example of $\mathfrak{m}_2$ in the case of $\mathbb{P}(1,1,2)$

Let us give an example of the composition $\mathfrak{m}_2$ in the case of $\mathbb{P}(1,1,2)$. For example, for $b-a=1$, $c-b=2$, let us consider the composition of the following generators.

$$\mathfrak{m}_2(V_{ab;(0,1,0)}, V_{bc;(0,0,1)}) = ?$$





$$\rightsquigarrow \mathfrak{m}_2(V_{ab;K_{ab}}, V_{bc;K_{bc}}) = e^{-(f_{ab;K_{ab}}(v_{ac;K_{ac}})+f_{bc;K_{bc}}(v_{ac;K_{ac}}))} V_{ac;K_{ac}}$$

The A_∞ product

- The same holds in the general case of $\mathbb{P}(q_0, \ldots, q_n)$ for $a < b < c$. Namely, we have

$$\mathfrak{m}_2(V_{ab;K_{ab}}, V_{bc;K_{bc}}) = e^{-(f_{ab;K_{ab}}(v_{ac;K_{ac}})+f_{bc;K_{bc}}(v_{ac;K_{ac}}))} V_{ac;K_{ac}},$$

  where $v_{ac;K_{ac}:=K_{ab}+K_{bc}}$ is a point on ∂P dividing the line segment $v_{ab;K_{ab}} v_{bc;K_{bc}}$ in the ratio $c - b : b - a$.
- $P \in Mo_{\mathcal{E}}(P)(L_a, L_a)$ forms the identity morphism.
- Remark. We see that for $V_{ab;K_{ab}} \in Mo_{\mathcal{E}}(L_a, L_b)$ and $V_{bc;K_{bc}} \in Mo_{\mathcal{E}}(L_b, L_c)$, the image $\gamma(T)$ of any $\gamma \in \mathcal{GT}(V_{ab;K_{ab}}, V_{bc;K_{bc}}; V_{ac;K_{ac}})$ is contained in the boundary ∂P unless $a = b = c$.

The Category on the complex side

On the complex side, we consider a DG category $DG(\mathbb{P}(q_0, \ldots, q_n))$ given as follows:

- Obj : $(\mathcal{O}(a), D_a),\ \ a \in \mathbb{Z}$
- Mor :

$$DG(\mathbb{P}(Q)(\mathcal{O}(a), \mathcal{O}(b))) = \Gamma(\mathcal{O}(a), \mathcal{O}(b)) \otimes_{C^\infty(\mathbb{P}(Q))} \Omega^{0,*}(\mathbb{P}(Q))$$

: Dolbeault resolution of $\mathrm{Hom}(\mathcal{O}(a), \mathcal{O}(b))$.

- composition and differential :

$$m(\psi_{ab}, \psi_{bc}) = \psi_{ab} \wedge \psi_{bc}, \quad d_{ab}(\psi) = 2(D_b^{(0,1)}\psi - (-1)^r \psi D_a^{(0,1)}),$$

where we decompose $D_a = D_a^{(1,0)} + D_a^{(0,1)}$, and $\psi \in DG^r(\mathbb{P}(Q))(\mathcal{O}(a), \mathcal{O}(b))$.

sketch of proof of the main theorem

In order to show that a DG-equivalence

$$Mo_{\mathcal{E}}(P) \simeq DG_{\mathcal{E}}(\mathbb{P}(q_0, \ldots, q_n))$$

...

- Here, recall that a DG-equivalence is a DG functor which induces a category equivalence on the corresponding cohomology categories.
- The objectwise correspondence between $Mo_{\mathcal{E}}(P)$ and $DG_{\mathcal{E}}(\mathbb{P}(Q))$ is induced by the SYZ transformation.
- To complete the proof, it is enough to show that there exists a quasi-isomorphism between the hom spaces at the cochain level and the compatibility of products structures holds.

The cohomologies $H^r(DG_{\mathcal{E}}(\mathbb{P}(Q))(\mathcal{O}(a), \mathcal{O}(b)))$

- For $\mathbb{P}(q_0, \ldots, q_n)$, let $S = \mathbb{C}[z_0, \ldots, z_n]$, where $\deg(z_i) = q_i$. We have the decomposition $S = \bigoplus_{r=0}^{\infty} S_r$. The graded piece S_r of degree r consists of homogeneous polynomials of degree r.

$$H^r(DG_{\mathcal{E}}(\mathbb{P}(Q))(\mathcal{O}(a), \mathcal{O}(b))) = \begin{cases} S_{b-a} & \text{for } r = 0,\ a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

- Example ($\mathbb{P}(1,1,2)$). $\deg(z_0) = \deg(z_1) = 1, \deg(z_2) = 2$. For $b - a = 2$, generators are $z_0^2,\ z_0 z_1,\ z_1^2,\ z_2$.
- Generators of S_{b-a} are

$$z^{K_{ab}} = z_0^{k_0} \cdots z_n^{k_n}$$

for $K_{ab} = (k_0, \ldots, k_n) \in (\mathbb{Z}_{\geq 0})^{n+1}$ with

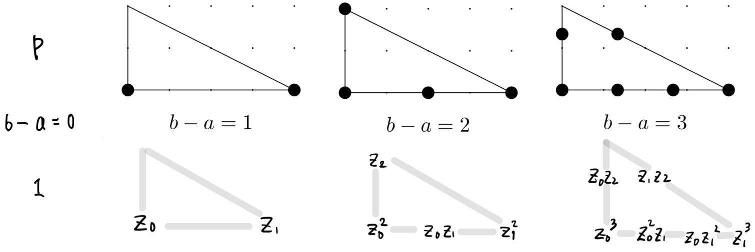
$$\sum^{n} q_i k_i = b - a.$$

correspondence

- We notice that both generators run over the same indices $K_{ab} \in (\mathbb{Z}_{\geq 0})^{n+1}$ with $\sum_{i=0}^{n} q_i a_i = b - a$. So there exists a one-to-one correspondence between the cohomologies.

$$V_{ab;K_{ab}} \longleftrightarrow z^{K_{ab}}$$

- In the case of $\mathbb{P}(1,1,2)$, $\mathcal{E} = (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3))$.



HMS for WPS

As a result of the main theorem, we obtain a version of homological mirror symmetry for weighted projective spaces.

Corollary (HMS for weighted projective spaces)

There exists an equivalence of triangulated categories

$$Tr(Mo_{\mathcal{E}}(P)) \simeq D^b(coh(\mathbb{P}(q_0, \ldots, q_n))),$$

where Tr denotes the construction of Bondal-Kapranov and Kontsevich.